IMPORTANT TECHNIQUES TO SOLVE FUNCTIONAL EQUATIONS

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ABSTRACT

Functional equations appear frequently in Mathematical Olympiads. In this article we focus on some types of functional equations and how to solve them.

Key words: Functional, equation, inequality, real number, positive, symmetric.

АННОТАЦИЯ

Функциональные уравнения часто появляются на математических олимпиадах. В этой статье мы сосредоточимся на некоторых типах функциональных уравнений и способах их решения

Ключевые слова: функциональный, уравнение, неравенство, настоящий номер, положительный, симметричный.

I.Switching

The main idea is to exchange the variables if one side of equation is symmetric with regard to the variables.

Problem 1.

Find all functions $f: R \to R$ such that, $f(f(x)+f(y))=f(x^2)+2x^2f(y)+f^2(y)$ holds for all $x, y \in R$ (1)

Solution: Let $p(x,y)$ denote the assertion (1). Left hand side is symmetric, by switching x, y we can see that

$$p(x,y): f(f(y)+f(x))=f(y^2)+2y^2f(x)+f^2(x) \qquad (2)$$

From (1) and (2) we get that:

$$f(x^2)+2x^2f(y)+f^2(y)=f(y^2)+2y^2f(x)+f^2(x)$$

By taking $y=0$ and $y=1$ we obtain that:

$$\begin{cases} f(x^2)+2x^2f(0)+f^2(0)=f(0)+f^2(x) \\ f(x^2)+2x^2f(1)+f^2(1)=f(1)+2f(x)+f^2(x) \end{cases} \quad (3)$$

From (3), we can easily get the answers $f(x)=0,\forall x\in R$ or $f(x)=x^2,\forall x\in R$

Problem 2.

Find all functions $f:R\to R$ such that,

$$f(x)+y+f(f(x)+y)=2f(x+y) \quad (4)$$

Solution: $f(f(x)+y)=2f(x+y)-f(x)-y$

$$p(x,f(y)):f(f(x)+f(y))=2f(x+f(y))-f(x)-f(y)= \\ =2(2f(x+y)-x-f(y))-f(x)-f(y) \Rightarrow$$

$$\Rightarrow f(f(x)+f(y))=4f(x+y)-3f(y)-f(x)-2x \quad (5)$$

By switching variables:

$$f(f(y)+f(x))=4f(y+x)-3f(x)-f(y)-2y \quad (6)$$

From (5) and (6) we can obtain :

$$f(x)-x=f(y)-y$$

By taking $y=0\Rightarrow f(x)=x+C$ by putting this solution to (4) we find $C=0$, then $f(x)=x$.

II. Flipped inequalities.

A general idea is to use two inequalitiesfor finding the value of f at a certain point. To this we will show $f(a)\geq c$ and $f(a)\leq c$, to obtain $f(a)=c$

In the following problem we will find $f(1)$ by this method.

Problem 3. Find all functions $f: R^{+} \to R^{+}$ such that for all positive real numbers x, y such that $x > y$ we have:

$$f(x-y) = f(x) - f(y) \cdot f(\frac{1}{x})y$$

Solution: We start the proof with the following claim.

Claim1: $\forall x \in R$ we have $f(x) \le x$.

Proof: Since $f(x-y) > 0$, we get that $f(x) - f(x) f(\frac{1}{x})y > 0$

$f(x)(1 - yf(\frac{1}{x}) > 0 \to f(\frac{1}{x}) < \frac{1}{y}$ for $x > y$.

By choosing y close enough to x we obtain that $f(\frac{1}{x}) \le \frac{1}{x} \Rightarrow f(x) \le x$

.

Claim 2. $f(1) = 1$

Proof. We know $f(1) \le 1$, so we must prove $f(1) \ge 1$.

Let $y < 1$. By plugging $x = 1$ we take

$$1 - y \ge f(1-y) = f(1) - f^{2}(1)y$$

$$1 - y \ge f(1) - f^{2}(1)y$$

$$(f(1) - 1)(y(1 + f(1)) - 1) \ge 0$$

We take $\forall y$ which $1 > y > \frac{1}{1 + f(1)}$ and we get $f(1) \ge 1 \Rightarrow f(1) = 1$

Claim 3. $\forall z < 1$ $f(z) = z$ we take $x = 1,$ $y = 1 - z$ $z < 1$

$$f(z) = 1 - (1 - z) = z$$

Now we take $y < x < 1$,

$$x - y = f(x-y) = f(x) - f(x) f(\frac{1}{x})y = x - xyf(\frac{1}{x})$$

$$\Rightarrow f(\frac{1}{x}) = \frac{1}{x} \text{ for } x > 1 \text{. Therefore } f(x) = x \text{ for } \forall x \in R$$

Overall that kind of problems are common in mathematical olympiads. We recommend to solve some functional equations by using this techniques.

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