INTEGRAL TENGSIZLIKLAR

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Annotatsiya: Bu maqolada olimpiada masalalari va ko‘plab muhim masalalarni hal qilishda ko‘p qo‘llaniladigan tengsizliklarni o‘rganamiz. Oldin ba’zi xossa va teoremalarni beramiz, keyin esa asosiy tengsizliklarni va ularni misollarda qanday qo‘llashni ko‘rib chiqamiz.

Kalit so‘zlar: Koshi-Shvarz tengsizligi, Minkovskiy tengsizligi,Gyolder tengsizligi,Chebishhev tengsizligi.

INTEGRAL INEQUALITIES

Abstract: In this article, we will study inequalities that are often used to solve olympiad problems and many important problems. First, we give some properties and theorems, and then we look a basic, important inequalities and how to use them in examples.

Key words: Cauchy-Schwarz inequality, Minkowski’s inequality, Hölder’s inequality, Chebyshev’s inequality.

Xossa 1. $f:[a,b]\to R$ funksiya Riman ma’nosida integrallanuvchi va nomanfiy funksiya bo‘lsin. U holda, $\int_a^b f(t)dt \geq 0$ tengsizlik o‘rinli bo‘ladi. (tenglik faqat va faqat f=0 b’lganda bajariladi.)

Teorema. Agar $f, g: [a, b] \to R$ funksiyalar uchun, $f \le g$ bo‘lsa, u holda $\int_a^b f(t)dt \le \int_a^b g(t)dt$ tengsizlik o‘rinli bo‘ladi. (Isboti yuqoridagi izhohdan osongina kelib chiqadi)

Xossa 2. Agar $f(x)$ $funksiya$ $[a; b]$ oraliqda integrallanuvchi bo‘lsa, u holda $|f(x)|$ funksiya ham shu oraliqda integrallanuvchi bo‘ladi va

$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$ tengsizlik o‘rinli bo‘ladi.

Torema (O‘rta qiymat haqidagi teorema): $f: [a, b] \to R$ uzluksiz funksiya va $g: [a, b] \to R$ nomanfiy integrallanuvchi funsiya bo‘lsin. U holda, shunday

$c \in [a, b]$ mavjudki, $\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$

tenglik o‘rinli bo‘ladi.

Teorema(Chegaranlaganlik). Agar $f: [a; b] \to R$ funksiya integrallanuvchi bo‘lsa, u holda $f -$ chegaralangan, $|f|$ integrallanuvchi va $\left|\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{1}{b-a}\int_a^b |f(t)|dt \le \sup_{a \le t \le b} |f(t)|$ bo‘ladi.

Lemma. Agar $f' - [a; b]$ da integrallanuvchi bo‘lsa, u holda

$0 \le \frac{1}{b-a}\int_a^b |f(x)|dx$-$\left|\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{3} \sup_{a \le x \le b} |f'(x)|$ tengsizlik o‘rinli bo‘ladi.

Lemma. Faraz qilaylik f- $[a; b]$ da uzluksiz differensiallanuvchi funksiya va $f(a) = f(b) = 0$ bo‘lsin. U holda

$\sup_{a \le t \le b} |f(t)| \le \frac{b-a}{2} \int_a^b |f'(t)|dt$ tengsizlik o‘rinli bo‘ladi.

Maxsus tengsizliklar

Koshi-Bunyakovskiy tengsizligi.

Agar $f(x)$ va $g(x)$ funksiyalar $[a; b]$ oraliqda integrallanuvchi bo‘lsa, u holda ushbu $f(x) - \alpha g(x)$ ($\alpha -$ ixtiyotiy o‘zgarmas son) funksiya ham $[a; b]$ oraliqda integrallanuvchi va

$\int_a^b \left(f(x) - \alpha g(x)\right)^2 dx \ge 0$ tengsizlik o‘rinli bo‘ladi (bu funksiyalarning kvadratlari ham integrallanuvchi).

Demak, ixtiyoriy o'zgarmas α son uchun

$$\alpha^2 \int_a^b g^2(x)dx - 2\alpha \int_a^b f(x)g(x)dx + \int_a^b f^2(x)dx \geq 0$$

tengsizlik o'rinli bo'ladi. Bu tengsizlikning chap tomonidagi ifoda α ga nisbatan kvadrat uchhad bo'lib, u α ning barcha haqiqiy qiymatlarida manfiy emas. Demak, bu kvadrat uchhadning diskriminanti musbat emas, ya'ni

$(\int_a^b f(x)g(x)dx)^2 - \int_a^b f^2(x)dx \int_a^b g^2(x)dx \leq 0$

Natijada,

$$\left|\int_a^b f(x)g(x)\,dx\right| \leq \sqrt{\int_a^b f^2(x)\,dx}\sqrt{\int_a^b g^2(x)dx}$$

tengsizlikka kelamiz. Bu tengsizlik Koshi-Bunyakovskiy tengsizligi deb ataladi.

Example 1 . Faraz qilaylik, $f: [a; b] \to R$ uzluksiz funksiya uchun $\int_0^1 f(x)dx = 1$ va $\int_0^1 xf(x)dx = 1$ tengliklar o'rinli bo'lsin. U holda, $\int_0^1 f^2(x)dx \geq 4$ tengsizlik o'rinli ekanligini isbotlang.

Isbot: Bu tengsizlikni isbotlash uchun Koshi-Shvarz tengsizligidan foydalana oloshimiz ko'rinib turibdi va berilgan shartlardan foydalanib $g(x)$ funksiya ko'rinishini tanlab olib bu funksiyani topamiz: $\int_0^1 f^2(x)dx \int_0^1 g^2(x)dx \geq \left(\int_0^1 f(x)g(x)dx\right)^2 = [g(x) = ax + b \ \ deb\ tanlab\ olishimiz\ mumkin\ ] \quad = (a + b)^2$ (masala shartidan shunga kelamiz) $(*)$

Endi $f(x) \sim g(x)$ deb olsak, u holda:

$\begin{cases} \int_0^1 (ax+b)dx = 1 \\ \int_0^1 x(ax+b)dx = 1 \end{cases}$ sistemaga kelamiz va bundan

$\begin{cases} \frac{a}{2} + b = 1 \\ \frac{a}{3} + \frac{b}{2} = 1 \end{cases} \quad => \quad \begin{cases} a = 6 \\ b = -2 \end{cases}$ larni topamiz.

Bundan esa,

$\int_0^1 g^2(x)dx = \int_0^1 (6x-2)^2 dx = 4$ ni hosil qilamiz.

Yuqoridagi $(*)$ dan $4\int_0^1 f^2(x)dx \geq 16$ => $\int_0^1 f(x)dx \geq 4$ bo‘ladi. Isbotlandi.

Example 2 . $f:[0;1] \to R$ uzluksiz hosilali differensiallanuvchi funksiya bo‘lib, $\int_0^1 f(x)dx = 1$ va $\int_0^1 xf(x)dx = 1$ bo‘lsin. U holda $\int_0^1 (f'(x))^2 dx \geq 30$ tengsizlik o‘rinli ekanligini isbotlang.

Isbot: Bu tengsizlikni ham Koshi-Shvarz tengsizligidan foydalanib isbotlay olamiz (buni masala shartidan va isbotlashimiz kerak bo‘lgan tengsizlikka qarab ko‘ra olamiz)

$\int_0^1 \left(f'(x)\right)^2 dx \int_0^1 g^2(x)dx \geq \left(\int_0^1 g(x)f'(x)dx\right)^2 = \Big(f(x)g(x)|_0^1 -$

$\int_0^1 g'(x)f(x)dx\Big)^2 = \begin{bmatrix} g(0)=0 \ ,g(1)=0 \\ g'(x)-chiziqli \end{bmatrix} =$

$\begin{bmatrix} g(x)=a(x^2-x) \\ g'(x)=2ax-a \end{bmatrix} = a^2$ bo‘ladi. $(*)$

$$\int_0^1 a^2(x^2-x)^2 dx = a^2\left(\frac{1}{5}-\frac{1}{2}+\frac{1}{3}\right) = \frac{a^2}{30}$$

$(*)$ dan $\frac{a^2}{30}\int_0^1 (f'(x))^2 dx \geq a^2$ => $\int_0^1 (f'(x))^2 dx \geq 30$ hosil qilamiz. Isbotlandi.

Example 3. $f:[0;1] \to R$ uzluksiz differensiallanuvchi funksiya bo‘lib, $f(0) = f(1) = 0$ bo‘lsin. U holda,

$\int_0^1 (f'(x))^2 dx \geq 12\left(\int_0^1 f(x)dx\right)^2$ tengsizlikni isbotlang.

Isbot: Xuddi yuqoridagi misollar kabi Koshi-Shvarz tengsizligiga ko‘ra

$(*)$ $\int_0^1 \left(f'(x)\right)^2 dx \int_0^1 \left(g(x)\right)^2 dx \geq \left(\int_0^1 f'(x)g(x)dx\right)^2 = \Big(f(x)g(x)|_0^1 -$

$\int_0^1 g'(x)f(x)dx\Big)^2 = \left(\int_0^1 g'(x)f(x)dx\right)^2 = \begin{bmatrix} g(x) = ax+b \\ g'(x) = a \end{bmatrix} =$

$a^2\left(\int_0^1 f(x)dx\right)^2$ hosil qilamiz.

$$\int_0^1 g^2(x)dx = \int_0^1 (ax+b)^2 dx = \frac{a^2}{3} + ab + b^2$$

$(*)$ ga ko'ra : $\int_0^1 \left(f'(x)\right)^2 dx \geq \frac{a^2}{\frac{a^2}{3}+ab+b^2}\left(\int_0^1 f(x)dx\right)^2$ bo'ladi. Endi buni masala shartidagi tengsizlikka tushurish uchun $\frac{a^2}{\frac{a^2}{3}+ab+b^2} = 12$ deb olishimiz kerak. Bundan $(a+2b)^2 = 0$ bo'ladi. Ya'ni $a = -2b$ bo'lishi kerak. Keling $a = 2$, $b = -1$ olamiz. Bundan

$g(x) = 2x - 1$ ekanligini kelib chiqadi. Buni $(*)$ ga qo'ysak va hisoblasak isbotlashimiz kerak bo'lgan tengsizlikka kelamiz. Isbotlandi.

Example 4. $f: [0;1] \to R$ uzluksiz differensiallanuvchi funksiya uchun $f\left(\frac{1}{2}\right) = 0$ bp'lsin. U holda $\int_0^1 (f'(x))^2 dx \geq 12(\int_0^1 f(x)dx)^2$ tengsizlikni isbotlang.

Isbot: Isbotlashimiz kerak bo'lgan tengsizlikdan ko'rinadiki Koshi-Shvarz tengsizligini qo'llashimiz mumkin.

1) $\left[0;\frac{1}{2}\right]$ uchun quyidagini yozamiz

$$\int_0^{\frac{1}{2}} \left(f'(x)\right)^2 dx \int_0^{\frac{1}{2}} g^2(x)dx \geq \left(\int_0^{\frac{1}{2}} f'(x)g(x)dx\right)^2$$

$$= \left(f(x)g(x)\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} f(x)g'(x)dx\right)^2$$

Endi $g(x)$ funksiya uchun quyidagicha shartlar berib uni topib olamiz (Ya'ni $g(x)$ ni tanlash o'zimizga bog'liq). Shuning uchun bizni tengsizligimizga tushadigan qilib, bu funksiya uchun ma'lum bir shartlar berish orqali isbotlashimiz kerak bo'lgan tengsizlikka keltirishga harakat qilamiz.

$\left[f(x)g(x)|_0^{\frac{1}{2}} = \right.$

$= 0 \quad va \quad g'(x)$

$- o'zgarmas\ bo'ladiganqilib\ tanlaymiz: \begin{matrix} g(0) = 0 \\ g'(x) - o'zgarmas \end{matrix}\ bundan\ g(x) =$

$\left. = x\ deb\ olishimiz\ mumkin \right]$

Yuqoridagi tengsizligimizdan va $g(x) = x$ ekanligidan quyidagini hosil qilamiz:

$$\frac{1}{24}\int_0^{\frac{1}{2}} (f'(x))^2 dx \geq \left(\int_0^{\frac{1}{2}} f(x)dx \right)^2 \leftarrow (1)$$

2) Endi $\left[\frac{1}{2}; 1\right]$ segment uchun yozamiz:

$$\int_{\frac{1}{2}}^{1} (f'(x))^2 dx \int_{\frac{1}{2}}^{1} g^2(x)dx \geq \left(\int_{\frac{1}{2}}^{1} g(x)f'(x)dx \right)^2$$

$$= \left(f(x)g(x)|_{\frac{1}{2}}^{1} - \int_{\frac{1}{2}}^{1} f(x)g'(x)dx \right)^2$$

Endi xuddi yuqoridagi kabi $\left[\begin{matrix} g(1) = 0 \\ g'(x) - o'zgarmas \end{matrix} => g(x) = 1 - x \right]$ deb olishimiz mumkin.

Yuqoridagi tengsizligimizga ko‘ra:

$$\frac{1}{24}\int_{\frac{1}{2}}^{1} (f'(x))^2 dx \geq \left(\int_{\frac{1}{2}}^{1} f(x)dx \right)^2 \leftarrow (2)$$

tengsizlikni hosil qilamiz.

$$(1)+(2)==\frac{1}{24}\int_0^1\left(f'(x)\right)^2dx\geq\left(\int_0^{\frac{1}{2}}f(x)\,dx\right)^2+\left(\int_{\frac{1}{2}}^1 f(x)dx\right)^2$$

$$\geq\frac{1}{2}\left(\int_0^{\frac{1}{2}}f(x)dx+\int_{\frac{1}{2}}^1 f(x)dx\right)^2$$

$$=\frac{1}{2}\left(\int_0^1 f(x)dx\right)^2\quad[(a^2+b^2)(c^2+d^2)$$

$$\geq(ac+bd)^2\ tengsizlikdan\ foydalanildi.]$$

Demak bundan, $\int_0^1(f'(x))^2dx\geq 12\left(\int_0^1 f(x)\,dx\right)^2$ bo‘ladi. Isbotlandi.

Example 5. $a)$ $f\in C^2[0;1]$ funksiya uchun quyidagi $f(0)=f(1)=0$ va $f'(1)=1$ shartlar bajarilsin. U holda $\int_0^1|f"(x)|^2dx$ ning eng kichik qiymatini toping. (Mustaqil yechish uchun)

$b)$ $f\in C^2[0;1]$ funksiya uchun quyudagi $f(0)=f(1)=0$ va $f'(0)]=1$ shartlar bajarilsin. U holda $\int_0^1|f"(x)|^2dx$ ning eng kichik qiymatini toping.

Isbot: Biz integral qiymatini kichraytirishimiz kerak va integral ostida funksiyaning kvadrati turibdi, demak biz boshqa biror funksiya topamiz va Koshi-Shvarz tengsizligidan foydalanib bu integral qiymatini kichraytiramiz:

$\int_0^1(f"(x))^2dx\int_0^1 g^2(x)dx\geq\left(\int_0^1 f"(x)g(x)dx\right)^2=\left(f'(x)g(x)|_0^1-\right.$

$\left.f(x)g'(x)|_0^1+\int_0^1 g"(x)f(x)dx\right)^2$

Shartga ko‘ra $f(x)g'(x)|_0^1=0$ bo‘ladi. Endi biz $g(x)$ funksiyani shuynday tanlashimiz kerakki, o‘ng tomonda faqat ozod son qolsin ($g(x)$ funksiyani o‘zimiz quramiz va $f(x)$ esa no‘malum). Shuning uchun o‘ng tomonda shartlardan tashqari f lik qismlarni 0 bo‘ladigan qilib $g(x)$ funksiyani tanlaymiz:

$g"(x)=0$ bo‘lsin. U holda qavs ichida faqat $f'(x)g(x)|_0^1=f'(1)g(1)-f'(0)g(0)$ da esa shartga ko‘ra $f'(0)=1$ va biz $g(x)$ funksiya uchun $g(1)=0$ deb olamiz. U holda, qavs ichida faqat $g(0)$ qoladi va bu shartlardan ko‘rinadiki $g(x) = a(x-1)$.

Bundan, $\int_0^1 (f"(x))^2 dx \int_0^2 a^2(x-1)^2 dx \geq a^2 \quad =>$

$\int_0^1 (f"(x))^2 dx \geq 3$ ekanligi kelib chiqadi. Endi tenglik

ham bajarilishini ko‘rsatishimiz kerak. Koshi-Shvarz tengsizligida tenglik ikki funksiya ekvivalent (teng) bo‘lganda bajariladi ya’ni $f''(x) \sim g(x)$ bo‘lishi kerak. Bundan, $f''(x) = a(x-1) \quad => \quad f(x) = \frac{a}{6}x^3 - \frac{a}{2}x^2 + +bx + c$ bo‘ladi. Endu shunday a, b, c o‘zgarmaslar mavjudmi yo‘qmi shuni tekshirishimiz kerak. Masala shartidan $\begin{cases} f(0) = 0 \\ f(1) = 0 \\ f'(0) = 1 \end{cases} \quad => \; f(x) = \frac{1}{2}x(x-1)(x-2)$

Demak tenglik ham bajarildi. Javob: 3

Example 6. $f(x)$ $-[0;1]$ da 2 marta differensiallanuvchu funksiya bo‘lib, $\int_0^1 f(x)dx = \frac{f(1)}{2}$ bo‘lsa,

$$\int_0^1 (f"(x))^2 dx \geq 30 f^2(0)$$

tengsizlikni isbotlang.

Isbot: Integral ostidagi funksiyada kvadrat daraja bor va integral qiymati kichraymoqda, bu Koshi-Shvarz tengsizligimizdan foydalanishimizga ishora va biz bu tengsizlikni Koshi-Shvarz tengsizligidan foydalanib isbotlashga harakat qilamiz:

$$(*) \qquad \int_0^1 (f"(x))^2 dx \int_0^1 g^2(x) dx$$

$$\geq \left(f'(x)g(x)|_0^1 - f(x)g'(x)|_0^1 + \int_0^1 g"(x)f(x)dx \right)^2$$

$$= \left(f'(1)g(1) - f'(0)g(0) - f(1)g'(1) + f(0)g'(0) \right.$$

$$\left. + \int_0^1 g"(x)f(x)dx \right)^2$$

Endi yuqoridagi masalalarda aytganimizdek, maslada berilgan shartlardan foydalana olishimiz va isbotlashimiz kerak bo'lgan tengsizlikka kelishimiz uchun $g(x)$ funksiyaga shartlar beramiz: $\begin{cases} g(1) = g(0) = g'(0) = 0 \\ g"(x) = 2g'(1) - o'zgarmas \end{cases}$ =>

$g(x) = g'(1)(x^2 - x)$ bundan, $g'(x) = g'(1)(2x - 1)$

bo'ladi => $g'(0) + g'(1) = 0$

Bulardan :

$$\int_0^1 g^2(x)dx = \int_0^1 (g'(1))^2(x^2 - x)^2 dx = \frac{(g'(1))^2}{30}$$

$$(*) : \frac{\left(g'(1)\right)^2}{30} \int_0^1 (f"(x))^2 dx \geq (g"(1))^2 f^2(0) \quad =>$$

$$\int_0^1 (f"(x))^2 dx \geq 30 f^2(0)$$ Isbotlandi.

Minkovskiy tengsizligi: Agar $f, g: [a; b] \to R$ va $p > 1$ bo'lsa, u holda quyidagi tengsizlik o'rinli:

$$\left(\int_a^b |f(x) + g(x)|^p \, dx\right)^{\frac{1}{p}} \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_a^b |g(x)|^p dx\right)^{\frac{1}{p}}$$

Gyolder tengsizligi: $f: [0; 1] \to R$ funkisya berilgan bo'lib, ixtiyoriy $p, q > 1$ lar uchun $\frac{1}{p} + \frac{1}{q} = 1$ bo'lsin. U holda quyidagi tengsizlik o'rinli:

$$\int_a^b |f(x)g(x)| dx \leq \left(\int_a^b |f(x)|^p\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q\right)^{\frac{1}{q}}$$

Koshi-Shvarz tengsizligi esa bu tengszilini $p = q = 2$ holi bo'ladi.

Example 7. $f: [0; 1] \to R^+$ integrallanuvchi funksiya bo'lsin. Isbotlang:

$$\int_0^1 f(x)dx \int_0^1 f^2(x)dx \leq \int_0^1 f^3(x)dx$$

Isbot: Gyolder tengsizligiga ko'ra:

$$p = 3, q = \frac{3}{2} : \left(\int_0^1 f^3(x)\right)^{\frac{1}{3}} \left(\int_0^1 (1)^{\frac{3}{2}}\right)^{\frac{2}{3}} \geq \int_0^1 f(x)dx \quad (1)$$

$p = \frac{3}{2}, q = 3 : (\int_0^1 (f^2(x))^{\frac{3}{2}} dx)^{\frac{2}{3}} (\int_0^1 1^3 dx)^{\frac{1}{3}} \geq \int_0^1 f^2(x)dx \qquad (2)$

(1) va (2) dan isbotlashimiz kerak bo‘lgan tengsizlik osongina kelib chiqadi.

Isbotlandi.

Chebishev tengsizligi: Ixtiyoriy $a < b$ haqiqiy sonlar va $f, g: [a; b] \to R$ bir xilda monoton (ya’ni bir vaqtda ikkalasi ham o‘suvchi yoki ikkalasi ham kamayuvchi) funksiyalar berilgan bo‘lsin. U holda

$$(b - a) \int_a^b f(x)g(x)dx \geq \left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right)$$

tengsizlik o‘rinli bo‘ladi.

Isbot: Berilgan oraliqda f va g – funksiyalar bir xilda monoton bo‘lgani uchun

$(f(x) - f(y))(g(x) - g(y)) \geq 0$ bo‘ladi.

Shuning uchun $[a; b] \times [a; b]$ da yuqorida aytgan 1-xossamizga ko‘ra:

$$\int_a^b \int_a^b (f(x) - f(y))(g(x) - g(y))dxdy \geq 0$$

bo‘ladi.

Bundan, $\int_a^b \int_a^b f(x)g(x)dxdy + \int_a^b \int_a^b f(y)g(y)dxdy - \int_a^b \int_a^b f(x)g(y)dxdy - \int_a^b \int_a^b f(y)g(x)dxdy \geq 0$ va bu esa

$(b - a) \int_a^b f(x)g(x)dx - (\int_a^b f(x)dx)(\int_a^b g(x)dx) \geq 0$ tengsizlikka ekvivalentdir.

Isbotlandi.

Example 8. $f: [0; 1] \to (0; +\infty)$ kamayuvchi funksiya berilgan bo‘lsin. Isbotlang:

$$\int_0^1 xf^2(x)dx \int_0^1 f(x)dx \leq \int_0^1 xf(x)dx \int_0^1 f^2(x)dx$$

Isbot: Masala shartiga ko‘ra f –funksiya kamayuvchi bo‘lganligi uchun $f(x)f(y)(f(x) - f(y))(x - y) \leq 0$ tengsizlik o‘rinli bo‘ladi. Endi 1-xossaga ko‘ra

$[a;b]\times[a;b]$ da : $\int_a^b\int_a^b f(x)f(y)\big(f(x)-f(y)\big)(x-y)dxdy \le 0$ tengsizlik bajariladi. Endi bunu ham xuddi Chebishev tengsizligining isbotidagi kabi ochib chiqib alohida-alohida yozib olib soddalashtirsak isbotlashimiz kerak bo‘lgan tengsizlikka kelamiz. Isbotlandi.

FOYDALANILGAN ADABIYOTLAR RO‘YXATI: (REFERENCES)

1. Otaboyev T.O‘. , Karimov J.A. , Raximov K.X. , Narzillayev N.X. , Rajabov S.M. , Karimov J.J. “Matematik analiz” fani bo‘yicha o‘quv-uslubiy qo‘llanma (2-semestr), 2016.

2. Razvan Gelca, Titu Andreescu “Putnam and Beyond”, 2007.

3. Constantin P. Niculescu, “An Introduction to Mathematical analysis” , Universitaria Press, Craiova, 2005.