BYURGERS TENGLAMASI VA UNING YECHIMI

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ANNOTATSIYA

Ushbu maqolada turli xil suyuqliklar va gazlar dinamikasiga oid Byurgers tipidagi tenglama va uning yechimlari o'rganilgan. Byurgers tipidagi tenglama uchun muammolar oilasi ba'zan bosimsiz ikki tezlikli gidrodinamika deb ataladi. Berilgan tenglama atroflicha o'ganib chiqildi. Tenglamaning yechimlari va grafigi hosil qilindi.

ABSTRACT

In this article, the Burgers-type equation for the dynamics of various liquids and gases and its solutions are studied. The family of problems for the Burgers-type equation is sometimes called pressureless two-velocity hydrodynamics. The given equation has been refined. The solutions and graph of the equation were generated.

KIRISH

Biz Burgers tenglamasining analitik murakkabligi birdan yuqori bo'lmagan barcha yechimlarini tasvirlaymiz. Ma'lum bo'lishicha, bunday echimlarning barchasi o'lchamlari 3 dan oshmaydigan va elementar funktsiyalar bilan ifodalanadigan to'rtta oilaga kiradi. Burgers 2-murakkablik tenglamasining yechimlari turkumiga misol keltirilgan.

Byurgers tenglamasi quyidagi kurinishga ega

$$z'_y + z * z'_x + z''_{xx} = 0 . \qquad (1)$$

$z = c(a(x) + b(y))$ kurinishda tanlab olinsin va bu erda (a,b,c) doimiy bo'lmagan analitik bir o'zgaruvchili funksiyalar bo'lsin. Tanlab olingan funksiyamizni (1) ga quysak quydagi kurinish xosil bo'ladi.

$$b_1c_1 + a_1c_0c_1 + a_1^2c_2 + a_2c_1 = 0$$

Bunda xar bir indeks shu funksiya xosilasining tartibi. Bunda a, b va c larning doimiy emasligi a_1, b_1 va c_1 larning aynan nol emasligini anglatadii. Ma'lumki

$$-\frac{c_2}{c_1} = \frac{a_1c_0 + a_2 + b_1}{a_1^2} \qquad (2)$$

(2) ning chap qismi $(a(x) + b(y))$ argumentli funksiya bo'lgani uchun, tenglikning o'ng qismi

$$L = b'(y)\frac{\partial}{\partial(x)} - a'(x)\frac{\partial}{\partial(x)} \qquad (3)$$

operatorning yadrosida yotishi kerak. Natijada quyidagi tengliikni hosil qilamiz:

$$a_1a_2b_1c_0 + a_1^2b_2 - a_1a_3b_1 + 2a_2^2b_1 + 2a_2b_1^2 = 0. \qquad (4)$$

1- holat. Agar $a_2 = 0$ bo‘lsa, (4) dan $b_2 = 0$ kelib chiqadi, ya’ni a va b chiziqli, aniqrog‘i $a(x) + b(y) = kx + ly + \gamma$. k va l nolga teng bo‘lmagan doimiylardir. Biz z ni $z = c(x + \lambda y)$ shaklida qidirmoqdamiz. Bunda (1) dan $\lambda c'(t) + c(t)c'(t) + c''(t) = 0$ natijani olamiz. Bu tenglamaning javobi

$$z = 2\mu\,\mathrm{th}\big(\mu(x + \lambda y + v)\big) - \lambda$$

2- holat. Agar $a_2 \neq 0$ bo‘lsa (4) dan ushbu natijani olamiz

$$c_0 = -\frac{a_1^2b_2 - a_1a_3b_1 + 2a_2^2b_1 + 2a_2b_1^2}{a_1a_2b_1} \qquad (5)$$

L operatorni qo‘llash orqali (5) dan quydagi tenglamani olamiz

$$a_1^4a_2b_1b_3 - a_1^4a_2b_2^2 + a_1^3a_3b_1^2b_2 + a_1^2a_2^2b_1^2b_2 + a_1^2a_2a_4b_1^3 - a_1^2a_3^2b_1^3 - 2a_1a_2^2a_3b_1^3 + 2a_2^4b_1^3 + 2a_2^3b_1^4 = 0. \qquad (6)$$

Bu munosabat, agar $b_2 = 0$ bo‘lsa, unda $a_2 -$ ham nol bo‘ladi, ya’ni ushbu holatda nafaqat a_2, balki b_2 ham noldan farq qiladi, degan xulosaga kelishimizga imkon beradi. c_0 uchun (5) ifodani (2) ifodadagi $\frac{c_2}{c_1}$ ni inobatga olgan holda yozsak quyidgi tenglika kelamiz:

$$a_1^4a_2b_1^2b_4 - 4a_1^4a_2b_1b_2b_3 + 3a_1^4a_2b_2^3 - a_1^4b_1^2b_2b_3 + a_1^4b_1b_2^3 + a_1^3a_3b_1^3b_3 - a_1^3a_3b_1^2b_2^2 + a_1^2a_2^2b_1^3b_3 - a_1^2a_2^2b_1^2b_2^2 - a_1^2a_2b_1^4b_3 - a_1^2a_2b_1^3b_2^2 + 2a_1a_2a_3b_1^4b_2 - 2a_2^3b_1^4b_2 - 2a_2^2b_1^5b_2 = 0. \qquad (7)$$

Ko‘rish mumkinki, bu munosabatlar (x, y) erkli o‘zgaruvchilarga yoki erkli funksiyalarga bog‘liq emas, balki faqat (a, b) ning hosilalarigagina bog‘liq. Endi $a_1 = A, a_2 = P(A), b_1 = B, b_2 = Q(B)$ belgilash orqali (7) tenglamadagi funksiyalarning differensial tartiblarini pasaytiramiz, ya’ni birinchi tartibli hosilalar yangi mustaqil o‘zgaruvchilar, ikkinchi tartibli hosilalar esa noma’lum funksiyalardir. Misol uchun $a_3 = P'(A)P(A)$. Shu bilan bir qatorda avvalgidek belgilashdan foydalanamiz $P' = p_1, P'' = p_2, \ldots, Q' = q_1, \ldots$. Shunda (6) va (7) munosabatlar quydagi ko‘rinishni oladi

$$A^2B^3p_0^2p_2 + A^4Bq_1q_0 + A^3B^2p_1q_0 - 2AB^3p_0^2p_1 - A^4q_0^2 + A^2B^2p_0q_0 + 2B^4p_0^2 + 2B^3p_0^3 = 0, \qquad (8)$$

$$-A^3B^3p_0^3p_1p_2 - A^5Bp_0p_1q_0q_1 + A^4B^2p_0^2p_2q_0 - A^4B^2p_0p_1^2q_0 + A^2B^4p_0^3p_2 + A^2B^3p_0^4p_2 + 2A^2B^3p_0^3p_1^2 + A^5Bq_0^2p_1 + A^5p_0q_0^2p_1 + A^4Bp_0^2q_1q_0 + A^3B^3p_0q_0p_1 - 4AB^4p_0^3p_1 - 4AB^3p_0^4p_1 - A^4Bp_0q_0^2 - A^4p_0^2q_0^2 + A^2B^3p_0^2q_0 - A^2B^2p_0^3q_0 + 2B^5p_0^3 + 4B^4p_0^4 + 2B^3p_0^5 = 0 \qquad (9)$$

$q_0 = b_2 \neq 0$ bo‘lgani uchun, (9) dan q_1 funksiya va bu funksiya A ga bog‘liq emasligini yozamiz, shunda

$(-A^4p_2 + 2A^2p_0)q_0 - A^3Bp_0^2p_3 - 2A^3Bp_0p_1p_2 + 4A^2Bp_0^2p_2 + 4A^2Bp_0p_1^2 - 4AB^2p_0p_1 - 12ABp_0^2p_1 + 8B^2p_0^2 + 8Bp_0^3 = 0.$ (10)

tenglamani hosil qilamiz. (10) tenglamaning quydagi ko‘rinishlarda ixchamlashimiz mumkin $Q = mB^2 + nB + l$ yoki $(-A^2p_2 + 2p_0) = 0$.

2.1 – holat. Q ifoda B ning kvadratiga bog‘liq, ya’ni $Q = mB^2 + nB + l$. Q ifodani (8) va (9) ga oborib qo‘yamiz. (8) ga tegishli B ga bog‘liq bo‘lmagan ozod hadni ajratamiz, natijada $l = 0$ ni hosil qilamiz. So‘ngra (9) dan B^5 qatnashgan hadni ajratib olib, u erdan p_1 ni p_0 orqali ifodalaymiz. Differensiallash orqali biz p_0 dan p_2 ga tegishli bo‘lgan ifodani olamiz. Bu ifodani (8) tenglkdagi B^4 qatnashgan hadga qo‘yib $A^2m\left(A^2m + 2P(A)\right) = 0$ tenglikni hosil qilamiz. Bizda quyidagi muqobil yechimlar bor: $m = 0$ (2.1.1 – holat) yoki $P(A) = -\frac{mA^2}{2}$ (2.1.2 holat) .

(2.1.1 – holat) (9) dan B^5 ning koefitsientini ajratsak $a_2 = P(A) = 0$ bo‘lib, bu 2-holat shartiga zid bo‘ladi.

(2.1.2 – holat) (8) va (9) shartlar $mn = 0$ zaruriy va yetarli shartni beradi. Lekin, $m = 0$ tenglik $a_2 = P(A) = 0$ ekanligini anglatadi, shu bois $n = 0$ va $Q(B) = mB^2$. (5) ifodani (P, Q) o‘zgaruvchilar orqali yozsak quyidagi tenglikka kelamiz

$$c_0 = -\frac{-Ap_1p_0B + A^2q_0 + 2p_0B^2 + 2p_0^2B}{Ap_0B}$$

Hosil qilingan P va Q larni qo‘ysak $c_0 = 0$ bo‘ladi. Ziddiyat.

(2.2 – holat) $(-A^2p_2 + 2p_0) = 0$. Ushbu tenglamani yechsak $P = mA^2 + n/A$ ko‘rinishidagi umumiy yechimga kelamiz. Bu yechimni (8) ifodaga qo‘yib, A qatnashmagan hadni ajratib olsak $n = 0$ ya’ni $P = mA^2$ hosil bo‘ladi. Buni esa (9) ifodaga qo‘yib, A^6 qatnashmagan hadni ajratsak va $m \neq 0$ ni inobatga olsak $Q = -mB^2$ yoki $Q = -2mB^2$ tenglikni hosil qilamiz.

(2.2.1 – holat) $Q = -mB^2$. Hosil qilingan oddiy differensial tenglmani yechish orqali quyidgi tengliklarni hosil qilamiz

$$a(x) = ln(x\alpha + \mu)\,\delta, \qquad b(y) = -\,ln(\beta y + \nu)\,\delta.$$

Bularni (5) ifodaga qo‘yish orqali esa quyidagi tenglikka kelamiz

$$z = c(a + b) = \frac{(x\alpha + \mu)\beta}{\alpha(\beta y + \nu)}.$$

(2.2.2 – holat) $Q = -2mB^2$. c_0 uchun olingan ifodaga $P(A) = mA^2$, $Q(B) = -2mB^2$ tengliklarni olib borib qo‘ysak $c_0 = 0$ ga eg bo‘lamiz. Ziddiyat

To‘la baho berish uchun faqat bitta o‘zgaruvchiga bog‘liq bo‘lgan (1) ning yechimni ham ko‘rib o‘tamiz. Bu bizning terminologiya[1]mizda murakkablik funktsiyalari nolga teng. $z = z(x)$ bo‘lsin, u holda (1) ifodadan $zz' + z'' = 0$ ko‘rinishidagi oddiy tenglama hosil bo‘ladi. Uni yechish orqali esa quyidagiga ega bo‘lamiz:

$$z(x) = \frac{1}{k} th\left(\frac{x+\alpha}{2k}\right).$$

(1) ning yechimi x ga bog‘liq bo‘lmasa, ayonki u oddiy o‘zgarmas bo‘ladi.

Demak quyidagi teoremani isbotladik:

Teorema 1.1

a) Byurgers tenglamasining ixtiyoriy $z = c(a(x) + b(y))$ ko‘rinishdagi yechimi (bunda $z_1(x, y)$ ikkala o‘zgaruvchiga ham bog‘liq) quyidagi ikkita oiladan biriga kiradi:

(I) $z_1 = 2\mu th\big(\mu(x + \lambda y + v)\big) - \lambda$

(II) $z_2(x, y) = \frac{x+\alpha}{y+\beta}$

b) Byurgers tenglamasining faqat bitta o‘zgaruvchiga bog‘liq bo‘lgan yechimi quyidagi ko‘rinishni bo‘ladi

(III) $z_3(x) = 2\mu th\big(\mu(x + \alpha)\big)$

(IV) $z_4(y) = \beta.$

Bunda $\lambda\mu \neq 0.$

Birinchi oila 3 parametrli, ikkinchi oila 2 parametrli, uchinchi oila 2 parametrli, to‘rtinchi oila 1 parametrli hisoblanadi.

Natija 2. Byurgers tenglamasining murakkablik darajasi 1 dan yuqori bo‘lmagan barcha yechimlari elementar funksiyalar bo‘ladi.

Byurgers tenglamasining murakkablik darajasi birdan yuqori bo‘lgan yechimlari ham mavjud.

Misol 3. (1) tenglamaning quyidagi yechimlar oilasini ko‘rib chiqamiz:

$$z = \frac{4x + 2A}{x^2 + Ax - 2y + B}.$$

Ko‘rinib turganidek, bu ifodaning murakkablik darajasi 2 dan yuqori emas. Biz uning birdan yuqori ekanligini ko‘rsatamiz. Buning uchun differensial operatorni hisoblaymiz:

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[1] Terminologiya – bir sohaga oid terminlar majmui

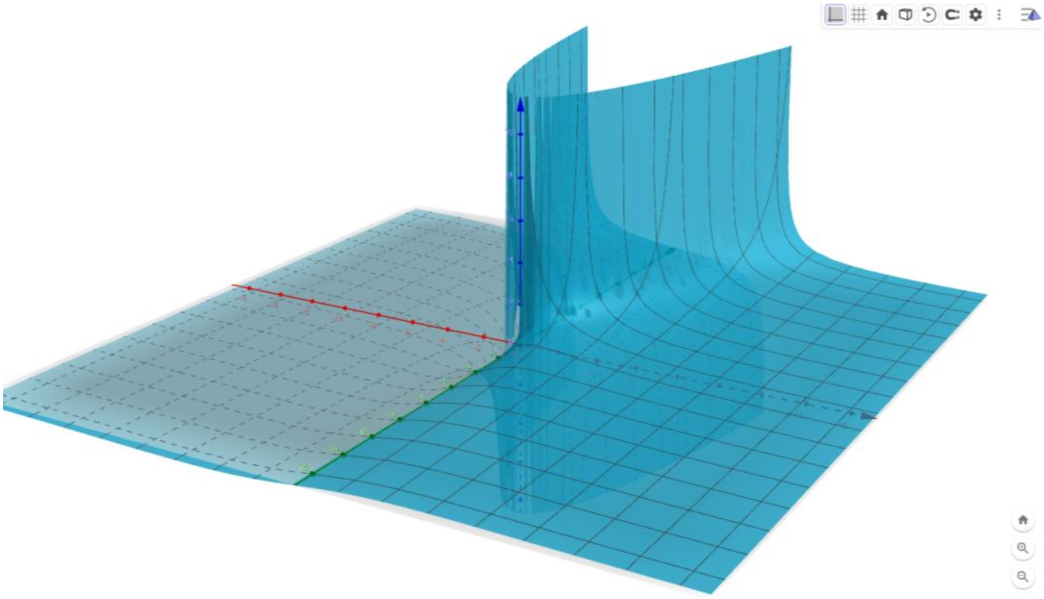
$$\delta(z) = \left(ln\left(\frac{z'_x}{z'_y}\right)\right)''_{xy} = -8\frac{A+2x}{(A^2+2Ax+2x^2-2B+4y)^2}$$

z funksiya faqat va faqat $\delta(z) \equiv 0$ bo‘lgandagina birdan yuqori bo‘lmagan murakkablikka ega bo‘ladi. Yuqoridagi natija esa buning aksini, ya’ni z ning murakkablik darajasi 1 dan yuqori ekanligini ko‘rsatadi. Demak uning murakkablik darjasi 2 ga teng. Shuningdek uning murakkablik darajasi A va B ning qiymatlariga bog‘liq bo‘lmagan holda 2 ga teng. Xususan, ushbu

$$z = \frac{4x}{x^2-2y}$$

funksiyaning ham murakkablik darajasi 2 ga teng bo‘ladi.

Funksiyaning grafigi quyidagi ko‘rinishda bo‘ladi.



FOYDALANILGAN ADABIYOTLAR RO‘YXATI: (REFERENCES)

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