BA’ZI TENGSIZLIKLARNI ISBOTLASHNING LAGRANJ KO‘PAYTUVCHILARI USULI

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ANNOTATSIYA

Ushbu maqolada o‘zgaruvchilariga bog‘liq funksiyasi ma’lum shartlarni qanoatlantiruvchi va o‘zgaruvchilarga bog‘liq qo‘shimcha shartlar bilan berilgan tengsizliklarni isbotlash usulini ko‘rib o‘tamiz. Bunda berilgan tengsizlik funksiyasi orqali yangi qo‘shimcha funksiya yaratish talab etiladi. Bunday tengsizliklarga [1] va [2] adabiyotlarda ham misollar keltirilgan.

Kalit so‘zlar: tengsizlik, funksiya, uzluksizlik, Lagranj funksiyasi, differensial, xususiy hosila, lokal ekstremum, Koshi tengsizligi.

THE METHOD OF LAGRANGE MULTIPLIERS FOR PROVING CERTAIN INEQUALITIES

ABSTRACT

In this article, we will consider the method of proving inequalities in which functions of their variables are continuous and differentiable and for those variables are given some conditions. In this given inequality, it is required to create a new additional function through the function. Examples of this kind of inequality are also given in the literature [1] and [2].

Keywords: inequality, function, continuity, Lagrange function, differential, partial derivative, local extremum, AM-GM inequality.

МЕТОД МНОЖИТЕЛЕЙ ЛАГРАНЖА ДЛЯ ДОКАЗАТЕЛЬСТВА НЕКОТОРЫЕ НЕРАВЕНСТВА

АННОТАЦИЯ

В данной статье мы рассмотрим метод доказательства неравенств в которых функции своих переменных непрерывны и дифференцируемы, а также для тех, переменным заданы некоторые условия. В данном неравенстве необходимо создать новая дополнительная функция через функцию неравенства. Примеры такого рода неравенстватакже приведены в литературе [1] и [2].

Ключевые слова: неравенство, функция, непрерывность, функция Лагранжа, дифференциал, частная производная, локальный экстремум, неравенство Коши.

Teorema. (Lagranj ko‘paytuvchilari teoremasi) $f(x_1,x_2,...,x_m)$ $I\subseteq\mathbb{R}^m$ ajralmaydigan sohada uzluksiz, differensiallanuvchi funksiya bo‘lsin, shuningdek $g_i(x_1,x_2,...,x_m)=0$, $i=1,2,...,k$ (bu yerda $k<m$) shartlar qanoatlantirilsin. U holda f funksiya I sohaning chetki nuqtalarida yoki $L=f-\sum_{i=1}^{k}\lambda_i g_i$ Lagranj funksiyasining xususiy hosilalari $(x_1,x_2,...,x_m$ bo‘yicha$)$ nolga teng nuqtalarida $g_i(x_1,x_2,...,x_m)=0$, $i=1,2,...,k$ shartlar bilan berilgan shartli ekstremumga erishadi [1, 177-b.].

Yuqoridagi teoremaga binoan tengsizliklarni isbotlashga doir bir qancha misollarni ko‘rib o‘tamiz:

1. $x,y,z\le 1$ haqiqiy sonlari $x+y+z=1$ tenglikni qanoatlantiradigan bo‘lsin. U holda quyidagi tengsizlikni isbotlang [1, 185-b.]:

$$\frac{1}{1+x^2}+\frac{1}{1+y^2}+\frac{1}{1+z^2}\le\frac{27}{10}$$

Isbot. $f(x,y,z)=\frac{1}{1+x^2}+\frac{1}{1+y^2}+\frac{1}{1+z^2}-\frac{27}{10}$ va $x+y+z=1$ dan foydalanib Lagranj funksiyasini tuzamiz:

$$L(x,y,z)=\frac{1}{1+x^2}+\frac{1}{1+y^2}+\frac{1}{1+z^2}-\lambda(x+y+z-1)$$

$f(x,y,z)$ funksiya $I\subseteq\mathbb{R}^3$ sohada uzluksiz va differensiallanuvchi. $L(x,y,z)$ funksiyaning xususiy hosilalarini topamiz:

$$\frac{\partial L}{\partial x}=-\frac{2x}{(1+x^2)^2}-\lambda \qquad \frac{\partial L}{\partial y}=-\frac{2y}{(1+y^2)^2}-\lambda \qquad \frac{\partial L}{\partial z}=-\frac{2z}{(1+z^2)^2}-\lambda$$

Teoremaga asosan $M(x,y,z)$ nuqtani topishimiz kerak, ya’ni M nuqtada L funksiyaning xususiy hosilalari nolga teng: $\frac{\partial L}{\partial x}(M)=\frac{\partial L}{\partial y}(M)=\frac{\partial L}{\partial z}(M)=0 \quad\Rightarrow$

$$-\frac{2x}{(1+x^2)^2}-\lambda=0 \qquad -\frac{2y}{(1+y^2)^2}-\lambda=0 \qquad -\frac{2z}{(1+z^2)^2}-\lambda=0$$

$$\lambda=-\frac{2x}{1+x^2}=-\frac{2y}{1+y^2}=-\frac{2z}{1+z^2}\ (1)\quad\Rightarrow\quad\frac{2x}{1+x^2}=\frac{2y}{1+y^2}=\frac{2z}{1+z^2}$$

$$x(1+y^2)^2=y(1+x^2)^2\ \Rightarrow\ (x-y)(xy(x^2+xy+y^2)+2xy-1)=0\ \Rightarrow$$

$x=y$ yoki $xy(x^2+xy+y^2)+2xy-1=0$

Xuddi shunday, quyidagi tengliklarga ega bo‘lamiz:

$y=z$ yoki $yz(y^2+yz+z^2)+2yz-1=0$

$x=z$ yoki $xz(x^2+xz+z^2)+2xz-1=0$

Izlanayotgan nuqtalarni topish uchun quyidagi holatlarni qarash maqsadga muvofiq bo‘ladi:

1) $x=y=z\ \Rightarrow\ x=y=z=\frac{1}{3}\ \Rightarrow\ \lambda=-\frac{2\cdot\frac{1}{3}}{\left(1+\frac{1}{9}\right)^2}=-\frac{27}{50}\ \Rightarrow\ M_1\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$

2) $x\neq y\neq z\ \Rightarrow\ \begin{cases}xy(x^2+xy+y^2)+2xy-1=0\\ yz(y^2+yz+z^2)+2yz-1=0\\ xz(x^2+xz+z^2)+2xz-1=0\end{cases}\ \Rightarrow$ ushbu sistemadagi tenglamalarni bir-biridan ayiramiz:

$$xy(x^2+xy+y^2)+2xy-1-yz(y^2+yz+z^2)-2yz+1=0\ \Rightarrow$$
$$\Rightarrow\ y(x-z)(x^2+y^2+z^2+xy+yz+xz+2)=0$$
$$xy(x^2+xy+y^2)+2xy-1-xz(x^2+xz+z^2)-2xz+1=0\ \Rightarrow$$
$$\Rightarrow\ x(y-z)(x^2+y^2+z^2+xy+yz+xz+2)=0$$
$$yz(y^2+yz+z^2)+2yz-1-xz(x^2+xz+z^2)-2xz+1=0\ \Rightarrow$$
$$\Rightarrow\ z(y-x)(x^2+y^2+z^2+xy+yz+xz+2)=0$$

(1) tenglik va $x+y+z=1$ bo‘yicha $x\neq 0$, $y\neq 0$, $z\neq 0$. Shuningdek, $x\neq y\neq z$ ligidan ko‘rishimiz mumkinki:

(2) $x^2+y^2+z^2+xy+yz+xz+2=0\ \Rightarrow$

$$2x^2+2y^2+2z^2+2xy+2yz+2xz=-4$$
$$(x+y)^2+(y+z)^2+(x+z)^2=-4$$

Yuqoridagi tenglama yechimga ega emasligidan, M_1 nuqta f funksiya shartli ekstremumga erishadigan yagona nuqta. Endi esa, f funksiya shu nuqtada eng katta qiymatiga yoki eng kichik qiymatiga erishishini aniqlash lozim. Buning uchun L Lagranj funksiyasining ikkinchi tartibli differensialining M_1 nuqtadagi ishorasini aniqlash zarur.

$$L(x,y,z)=\frac{1}{1+x^2}+\frac{1}{1+y^2}+\frac{1}{1+z^2}+\frac{27}{50}(x+y+z-1)$$

$$dL = -\frac{2x}{(1+x^2)^2}dx - \frac{2y}{(1+y^2)^2}dy - \frac{2z}{(1+z^2)^2}dz + \frac{27}{50}(dx+dy+dz)$$

$$d^2L = -2\left(\frac{1-3x^2}{(1+x^2)^3}dx^2 + \frac{1-3y^2}{(1+y^2)^3}dy^2 + \frac{1-3z^2}{(1+z^2)^3}dz^2\right)$$

$$d^2L(M_1) = -2\cdot\frac{2}{3}\cdot\frac{9^3}{10^3}(dx^2+dy^2+dz^2) = -\frac{243}{250}(dx^2+dy^2+dz^2) < 0$$

Demak, M_1 nuqta f funksiya shartli lokal maksimumga erishadigan nuqta.

$$f(M_1) = f\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) = \frac{1}{1+\frac{1}{3^2}} + \frac{1}{1+\frac{1}{3^2}} + \frac{1}{1+\frac{1}{3^2}} = \frac{27}{10} \quad \Rightarrow \quad f(x,y,z) \le \frac{27}{10}$$

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2} \le \frac{27}{10} \qquad \blacksquare$$

2. $a,b,c > 0$ haqiqiy sonlar hamda $a^2+b^2+c^2=3$ bo‘lsin. U holda, quyidagi tengsizlikni isbotlang [1, 185-b.]:

$$a^3(b+c) + b^3(a+c) + c^3(a+b) \le 6$$

Isbot. $f(a,b,c) = a^3(b+c)+b^3(a+c)+c^3(a+b)$ va $a^2+b^2+c^2=3$ ekanligidan foydalanib Lagranj funksiyasini tuzib olamiz:

$$L(a,b,c) = a^3(b+c)+b^3(a+c)+c^3(a+b) - \lambda(a^2+b^2+c^2-3)$$

Endi esa, L funksiyasi uchun xususiy hosilalarni topib, nolga tenglaymiz:

$$\frac{\partial L}{\partial a} = 3a^2(b+c)+b^3+c^3-2a\lambda \quad \Rightarrow \quad 3a^2(b+c)+b^3+c^3-2a\lambda = 0$$

$$\frac{\partial L}{\partial b} = a^3+3b^2(c+a)+c^3-2b\lambda \quad \Rightarrow \quad a^3+3b^2(c+a)+c^3-2b\lambda = 0$$

$$\frac{\partial L}{\partial c} = a^3+b^3+3c^2(a+b)-2c\lambda \quad \Rightarrow \quad a^3+b^3+3c^2(a+b)-2c\lambda = 0$$

$$\lambda = \frac{3a^2(b+c)+b^3+c^3}{2a} = \frac{a^3+3b^2(c+a)+c^3}{2b} = \frac{a^3+b^3+3c^2(a+b)}{2c}$$

$$(3a^2(b+c)+b^3+c^3)b = (a^3+3b^2(c+a)+c^3)a$$

$$3a^2b^2+3a^2bc+b^4+bc^3 = a^4+3b^2ca+3b^2a^2+c^3a$$

$$(b-a)(-3abc+c^3+(b+a)(b^2+a^2)) = 0$$

Bundan ko‘rishimiz mumkinki, $a=b$ yoki $(a^2+b^2)(a+b)+c^3=3abc$.

$$(a^2+b^2)(a+b) > (a^2-ab+b^2)(a+b)$$

$$(a^2+b^2)(a+b) > a^3+b^3$$

Koshi tengsizligi bo‘yicha quyidagi natijalarga ega bo‘lamiz:

$$(a^2+b^2)(a+b)+c^3 > a^3+b^3+c^3 \ge 3abc \quad \Rightarrow \quad (a^2+b^2)(a+b)+c^3 \ne 3abc$$

Xuddi shunday, $b=c$ va $a=c$ tengliklari kelib chiqadi. Demak, $a=b=c=1$.

$\lambda=\dfrac{3\cdot 2+1+1}{2}=4$ $M(1,1,1)$ $\Rightarrow$

$L(a,b,c)=a^3(b+c)+b^3(c+a)+c^3(a+b)-4(a^2+b^2+c^2-3)$

$dL=(3a^2(b+c)+b^3+c^3-8a)da+(a^3+3b^2(c+a)+c^3-8b)db+(a^3+b^3+3c^2(a+b)-8c)dc$

$d^2L=(6a(b+c)da+3a^2(db+dc)+3b^2db+3c^2dc-8da)da+$

$+(3a^2da+6b(c+a)db+3b^2(dc+da)+3c^2dc-8db)db+$

$+(3a^2da+3b^2db+6c(a+b)dc+3c^2(da+db)-8dc)dc$

$$d^2L(M)=4da^2+4db^2+4dc^2+12dadb+12dbdc+12dadc$$

$a^2+b^2+c^2=3 \Rightarrow 2ada+2bdb+2cdc=0$

M nuqta uchun $da+db+dc=0 \Rightarrow da=-(db+dc)$.

$d^2L(M)=4(db+dc)^2+4db^2+4dc^2+12(db+dc)(-db-dc)+12dbdc \Rightarrow$

$d^2L=-4(db^2+dbdc+dc^2)<0$

Bundan ko‘rinib turibdiki, M nuqta lokal maksimum va $f(a,b,c)\le f(M)$.

$f(1,1,1)=1\cdot 2+1\cdot 2+1\cdot 2=6 \Rightarrow a^3(b+c)+b^3(c+a)+c^3(a+b)\le 6$ ■

3. $a,b,c>0$ haqiyqiy sonlar hamda $a+b+c=1$ bo‘lsin. U holda quyidagi tengsiz-
likni isbotlang [2, 9-b.]:

$5(a^2+b^2+c^2)\le 6(a^3+b^3+c^3)+1$

Isbot. $f(a,b,c)=5(a^2+b^2+c^2)-6(a^3+b^3+c^3)$ va $a+b+c=1$ ligidan foydalanib Lagranj funksiyasini tuzib olamiz:

$L(a,b,c)=5(a^2+b^2+c^2)-6(a^3+b^3+c^3)-\lambda(a+b+c-1)$

L funksiyaning xususiy hosilalarini nolga tenglaymiz:

$\dfrac{\partial L}{\partial a}=10a-18a^2-\lambda \Rightarrow 10a-18a^2-\lambda=0$

$\dfrac{\partial L}{\partial b}=10b-18b^2-\lambda \Rightarrow 10b-18b^2-\lambda=0$

$\dfrac{\partial L}{\partial c}=10c-18c^2-\lambda \Rightarrow 10c-18c^2-\lambda=0$

$$\lambda=10a-18a^2=10b-18b^2=10c-18c^2$$

$10(a-b)=18(a-b)(a+b) \Rightarrow a=b$ yoki $a+b=\dfrac{5}{9} \Rightarrow c=\dfrac{4}{9}$

Xuddi shunday:

$b=c$ yoki $a=\dfrac{4}{9}$ va $a=c$ yoki $b=\dfrac{4}{9}$

Quyidagi holatlarni qaraymiz:

1) $a=b=c=\dfrac{1}{3} \Rightarrow M_1\left(\dfrac{1}{3},\dfrac{1}{3},\dfrac{1}{3}\right) \Rightarrow \lambda_1=\dfrac{4}{3}$

2) $M_2\left(\frac{1}{9},\frac{4}{9},\frac{4}{9}\right)$, $M_3\left(\frac{4}{9},\frac{1}{9},\frac{4}{9}\right)$, $M_4\left(\frac{4}{9},\frac{4}{9},\frac{1}{9}\right)$ nuqtalar uchun λ ning qiymati bir xil:

$$\lambda_2 = 10a - 18a^2 = 10b - 18b^2 = 10c - 18c^2 = \frac{8}{9}$$

$$L(a,b,c) = 5(a^2+b^2+c^2) - 6(a^3+b^3+c^3) - \lambda(a+b+c-1)$$

$$dL = 10ada + 10bdb + 10cdc - 18a^2da - 18b^2db - 18c^2dc - \lambda da - \lambda db - \lambda dc$$

$$d^2L = (10-36a)da^2 + (10-36b)db^2 + (10-36c)dc^2$$

$M_1\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) \Rightarrow d^2L(M_1) = -2(da^2+db^2+dc^2) < 0 \Rightarrow M_1$ nuqta lokal maksimum.

$$f(M_1) = 5\cdot\left(\frac{1}{9}+\frac{1}{9}+\frac{1}{9}\right) - 6\cdot\left(\frac{1}{27}+\frac{1}{27}+\frac{1}{27}\right) = \frac{5}{3}-\frac{2}{3} = 1 \Rightarrow f(a,b,c) \le f(M_1)$$

Endi esa, boshqa nuqtalarni ham ko‘rib chiqamiz:

$a+b+c=1 \Rightarrow da+db+dc=0$. M_2 nuqta uchun: $d^2L(M_2) = 6da^2 - 6db^2 - 6dc^2 = 12dbdc$ $\Delta = \begin{vmatrix} 0 & 6 \\ 6 & 0 \end{vmatrix} = -36 < 0 \Rightarrow M_2$ nuqta uchun funksiya shartli ekstremumga ega emas

[3, 106-b.].

Xuddi shunday, M_3, M_4 nuqtalarda ham shartli ekstremum mavjud emas.

Demak, M_1 nuqta yagona ekstremumga erishiladigan nuqta va quyidagi tengsizlik o‘rinli:

$$5(a^2+b^2+c^2) \le 6(a^3+b^3+c^3)+1 \qquad \blacksquare$$

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