GIPERBOLIK TENGLAMALAR SISTEMASI UCHUN KOSHI MASALASI VA CHEGARAVIY MASALA

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ANNOTATSIYA

Mazkur maqolada matematik fizikaning birinchi tartibli giperbolik sistema uchun qo‘yiladigan Koshi masalasi va chegaraviy masala qoyilishi hamda ayrim misollar orqali masalani yechish tahlil qilinadi.

Kalit so‘zlar: Giperbolik tenglamalar sistemasi. Koshi masalasi. Chegaraviy masala. Boshlang‘ich shart. Chegraviy shart. Xarakteristika.

KIRISH

Fizika, mexanika, biologiya, ekologiya va boshqa sohalarning ko‘pgina masalalarining matematik modellari birinchi tartibli kvazichiziqli tenglamalar uchun turli ko‘rinishdagi masalalarni o‘ganishga keltiriladi. Hozirgi kunga kelib bu fan sohasi mustaqil fundamental fanlar qatoriga qo‘shilgan bo‘lib, keng miqiyosdagi ilmiy-amaliy izlanishlar natijasida ko‘plab tabiiy jarayonlarning matematik modellari xususiy hosilali differensial tenglamalar yordamida xususan, birinchi tartibli tenglamalar va giperbolik sistemalar yordamida tuzilmoqda.

Birinchi tartibli giperbolik tenglamalar sistemasi to‘lqin xarakteridagi tabiiy jarayonlarni, suyuqlik va gazlarning muhitda tarqalish qonuniyatlarini, suyuqlik va gazlarda jismlar harakati qonuniyatlarini, tovush to‘lqinlarining tarqalish qonuniyatlarini va boshqa fizik, gidrodinamik, aerodinamik, gazodinamik jarayonlarni o‘rganishga, modellarini tuzishga keng tadbiq qilinadi. Demak, birinchi tartibli xususiy hosilali differensial tenglamalar tabiatini o‘rganish hamda ularga asoslangan holda giperbolik tenglamalar sistemasi mavzusini o‘rganish o‘zining keng ko‘lamli nazariy va amaliy ahamiyatiga ega ekan.

ASOSIY QISM

Ko‘pincha ikki erkli o‘zgaruvchili birinchi tartibli xususiy hosilali differensial tenglamalar nazariyasida 1-erkli o‘zgaruvchi sifati masofa hamda 2-erkli o‘zgaruvchi sifatida vaqt qaraladigan $u(x,t)$ funksiya va uning xususiy hosilalari qatnashgan tenglamalar hamda tenglamalar sistemasi o‘rganiladi. Biz bu ishda $x \to t$ va $y \to x$ kabi almashtirish yordamida tenglama uchun qo‘yiladigan chegaraviy masalalar o‘rganamiz. Demak, bir o‘lchamli fazoda vaqtga bog‘liq o‘zgaruvchi noma’lum $u(x,t)$

funksiya va uning o‘zgaruvchilar bo‘yicha xususiy hosilalari qatnashgan tenglamani o‘rganamiz .

Bizga $G \in R^2$ sohada vektor ko‘rinishidagi quyidagi sistema berilgan bo‘lsin

$$u_t + Du_x + u = f. \qquad (1)$$

(1) sistemaning quyidagi

$$u(x, o) = \begin{pmatrix} u_1(x,0) \\ u_2(x,0) \end{pmatrix} = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \qquad (2)$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimini topish masalasiga tenglamalar sistemasi uchun boshlang‘ich shartli masala yoki Koshi masalasi deyiladi.

(1)-(2) boshlang‘ich masalaning yechimini xarakteristikalar metodi yordamida topamiz. Agar D matritsa

$$D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}, \ |D| \neq 0, \ DD^{-1} = D^{-1}D \qquad (3)$$

shartni qanoatlantirsa, u holda (1) sistema

$$\begin{cases} w_{1t} + \lambda_1 w_{1x} + w_1 = \tilde{f_1} \\ w_{2t} + \lambda_2 w_{2x} + w_2 = \tilde{f_2} \end{cases} \qquad (4)$$

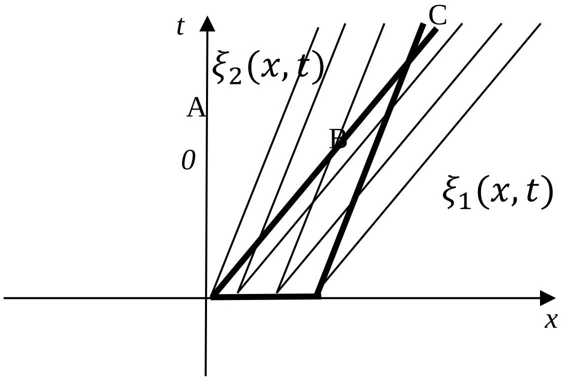
ko‘rinishdagi tenglamalari ajralgan yangi sistema bilan teng kuchli almashtiriladi. So‘ngra

$$u_t + Du_x + u = \tilde{f} \qquad (5)$$

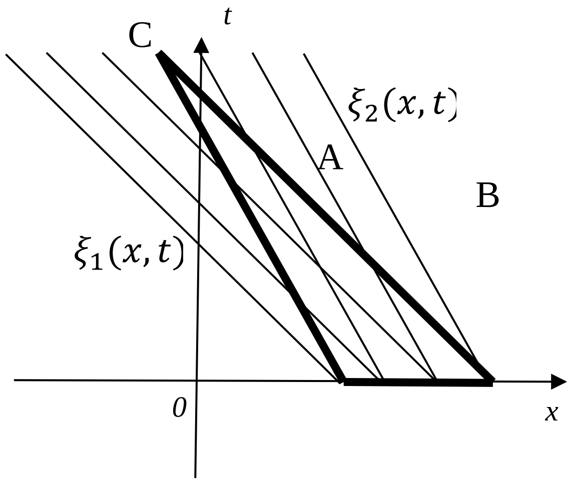
(5) formula va (4) munosabatlardan hamda

$$w(x, 0) = P^{-1}u(x, 0) \qquad (6)$$

boshlang‘ich shartdan foydalanib, w vektor funksiyaga nisbatan yangi sistema uchun Koshi masalasining yechimlarini topib $u = Pw$ alamashtirish orqali talab qilingan (1)-(2) masala yechimiga ega bo‘lamiz.



1 -chizma. $\lambda_2 > \lambda_1 >$



2 -chizma. $\lambda_1 < \lambda_2 < 0$

(1)-(2) Koshi masalasining yechimi ΔABC xarakteristik uchburchakda bir qiymatli topiladi (1-,2-chizmalar) [5,6,9].

1-Misol. Quyidagi

$$u_t - \begin{pmatrix} 1 & 0 \\ 5 & 3 \end{pmatrix} u_x = 0,$$

sistema uchun

$$u(x,0) = \begin{pmatrix} u_1(x,0) \\ u_2(x,0) \end{pmatrix} = \begin{pmatrix} e^{iax} \\ 0 \end{pmatrix}, a \in R$$

boshlang‘ich shartni qanoatlantiruvchi Koshi masalasining yechimini toping.

Yechish: $A = \begin{pmatrix} 1 & 0 \\ 5 & 3 \end{pmatrix}$ matritsaning xos sonlari $\lambda_1 = -1$, $\lambda_2 = -3$ larga teng bo‘lib, ularga mos kelgan xos vektorlar $\vec{e}_1 = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ lardan iborat. U holda

$$P = \begin{pmatrix} 2 & 0 \\ -5 & 1 \end{pmatrix}, \qquad P^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{5}{2} & 1 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$$

topiladi. $u = Pw$ belgilashni kiritib, quyidagicha ketma-ketlikdagi amallar yordamida yangi giperbolik sistemaga ega bo‘lamiz, ya’ni

$u_t + Au_x = 0,$

$Pw_t + APw_x = 0,$

$$w_t + P^{-1}APw_x = 0,$$
$$w_t + \Lambda w_x = 0.$$

Demak, yangi

$$w_t + \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} w_x = 0,$$

sistemani boshlang‘ich shart bilan birgalikda olib

$$w(x,0) = P^{-1}u(x,0) = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{5}{2} & 1 \end{pmatrix} \begin{pmatrix} e^{iax} \\ 0 \end{pmatrix} = \frac{1}{2} e^{iax} \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

quyidagicha ikkita alohida masalaga kelamiz

$$\begin{cases} w_{1t} - w_{1x} = 0 \\ w_1(x,0) = \frac{1}{2} e^{iax} \end{cases} \quad \text{va} \quad \begin{cases} w_{2t} - 3w_{2x} = 0 \\ w_1(x,0) = \frac{5}{2} e^{iax} \end{cases}.$$

Yuqoridagi ikkita Koshi masalasini xarakteristikalar usulida yechib

$w_1(x,t) = \frac{1}{2} e^{ia(x+t)}$ va $w_2(x,t) = \frac{5}{2} e^{ia(x+3t)}$ larga ega bo‘lamiz.

U holda

$$u = Pw = \begin{pmatrix} 2 & 0 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2}e^{ia(x+t)} \\ \frac{5}{2}e^{ia(x+3t)} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}e^{ia(x+t)} \\ -\frac{5}{2}e^{ia(x+t)} + \frac{5}{2}e^{ia(x+3t)} \end{pmatrix}$$

talab qilingan masala yechimi topiladi, demak Koshi masalasining yechimi

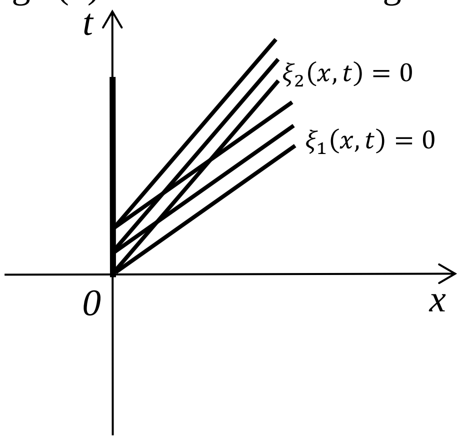
$$u(x,t) = \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{ia(x+t)} \\ -\frac{5}{2}e^{ia(x+t)} + \frac{5}{2}e^{ia(x+3t)} \end{pmatrix}$$ dan iborat ekan.

Chegaraviy masala.

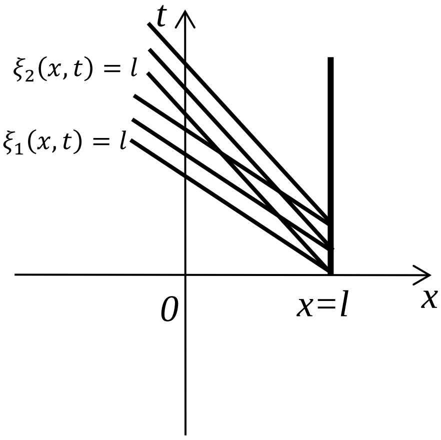
Bizga $D = \{(x,t): 0 \leq x,t < +\infty\}$ sohada (1) sistemaning quyidagi

$$u_1(0,t) = \psi_1(t),\ \ u_2(l,t) = \psi_2(t) \qquad (7)$$

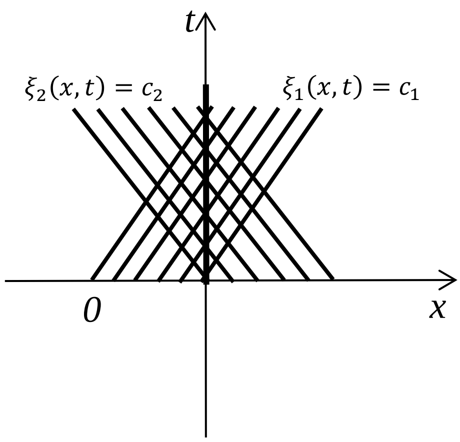
chegaraviy shartni qanoatlantiruvchi yechimini topish talab qilinsin. U holda bu masalasiga (1) sistema uchun chegaraviy masala deyiladi.



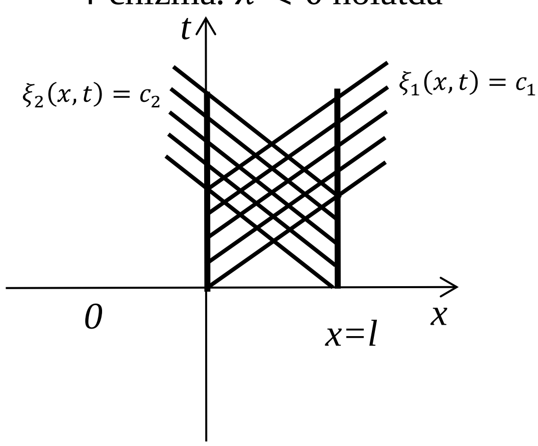
3-chizma. $\lambda > 0$ holatda



4-chizma. $\lambda < 0$ holatda



5-chizma.$\lambda_1 > 0, \lambda_2 < 0$ holatda



6-chizma.$\lambda_1 > 0, \lambda_2 < 0$ holatda

Chegaraviy masalalar qo‘yilishida $\lambda_{1,2}$ xarakteristik sonlarning ishorasi muhim rol o‘ynaydi.

Agar D normal matritsa bo‘lsa, ya’ni (3) shartlarni qanoatlantirsa u holda (4)-(5) formulalar yordamida, (1) sistemani tenglamalari ajralgan

$$\begin{cases} w_{1t} + \lambda_1 w_{1x} + w_1 = \widetilde{f_1} \\ w_{2t} + \lambda_2 w_{2x} + w_2 = \widetilde{f_2} \end{cases} \quad (7)$$

ko‘rinishidagi w vektor funksiyaga nisbatan yangi sistemaga almashtirish mumkin bo‘ladi. (4)-(5) formulalardan foydalanib w vektor funksiyaning umumiy yechimini topib

$$\begin{pmatrix} w_1(x=0,t) \\ w_2(x=l,t) \end{pmatrix} = P^{-1} \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} \quad (8)$$

shart bilan birgalikda chegaraviy masalaning yechimi topiladi hamda $u_1(x,t)$, $u_2(x,t)$ larga nisbatan (1)-(2) masala yechimiga ega bo‘lamiz.

Ma’lumki, saqlanish qonunlarining giperbolik sistemalarni yechish usullari gazadinamika masalalariga keng qo‘llaniladi. Quyida gazning truba orqali harakatini modellashtiruvchi ikki fazali masala qaraymiz.

Masalaning qo‘yilishi: $D_1 = \{(x,t): -\infty < x < 0\,, t > 0\}$ va $D_2 = \{(x,t): 0 < x < +\infty\,, t > 0\}$ sohalarda

$$\begin{cases} u_{1t}(x,t) + a_1 v_{1x}(x,t) + b_1 u_1(x,t) = f_1(x,t) \\ v_{1t}(x,t) + a_1 u_{1x}(x,t) + b_1 v_1(x,t) = g_1(x,t) \end{cases} \quad (9)$$

$$\begin{cases} u_{2t}(x,t) + a_2 v_{2x}(x,t) + b_2 u_2(x,t) = f_2(x,t) \\ v_{2t}(x,t) + a_2 u_{2x}(x,t) + b_2 v_2(x,t) = g_2(x,t) \end{cases} \quad (10)$$

sistemalarni va

$$u_1(x,0) = \phi_1(x)\,, v_1(x,0) = \psi_1(x)\,, -\infty < x \le 0 \quad (11)$$

$$u_2(x,0) = \phi_2(x)\,, v_2(x,0) = \psi_2(x)\,, 0 < x \le +\infty \quad (12)$$

boshlang‘ich hamda

$$u_1(0,t) - v_1(0,t) = \mu(t),\ t \ge 0 \quad (13)$$

$$u_2(0,t) + v_2(0,t) = \mu(t),\ t \ge 0 \quad (14)$$

chegaraviy shartlarni qanoatlantiruvchi $u_i(x,t)$, $v_i(x,t)$ noma’lum funksiyalar topilsin, $i = 1,2$. Bu yerda $f_i(x,t), g_i(x,t), \phi_i(x), \psi_i(x), \mu(t)$ - berilgan uzluksiz differensiallanuvchi funksiyalar, a_i, b_i o‘zgarmas sonlar, $a_i > 0, b_i > 0$, $i = 1,2$.

Yechish: Masalani yechish uchun quyidagicha yangi funksiyalar kiritamiz $i = 1,2$

$$\begin{cases} U_i(x,t) = u_i(x,t) + v_i(x,t) \\ V_i(x,t) = u_i(x,t) - v_i(x,t) \end{cases} \quad (15)$$

$$\begin{cases} F_i(x,t) = f_i(x,t) + g_i(x,t) \\ G_i(x,t) = f_i(x,t) - g_i(x,t). \end{cases} \quad (16)$$

$$\Phi_i(x) = \phi_i(x) + \psi_i(x), \Psi_i(x) = \phi_i(x) - \psi_i(x) \quad (17)$$

Natijada (15), (16) va (17) almashtishlardan keyin (9)-(14) masala quyidagicha ko‘rinishga keladi

$$\begin{cases} U_{1t}(x,t) + a_1 U_{1x}(x,t) + U_1 u_1(x,t) = F_1(x,t) \\ V_{1t}(x,t) - a_1 V_{1x}(x,t) + b_1 V_1(x,t) = G(x,t) \end{cases} \quad (18)$$

$$U_1(x,0) = \Phi_1(x)\,,\ V_1(x,0) = \Psi_1(x)\ ,\ 0 < x \le +\infty \quad (19)$$

$$V_1(0,t) = \mu(t), t \ge 0 \quad (20)$$

$$\begin{cases} U_{2t}(x,t) + a_2 U_{2x}(x,t) + b_2 U_2(x,t) = F_2(x,t) \\ V_{2t}(x,t) - a_2 V_{2x}(x,t) + b_2 V_2(x,t) = G_2(x,t) \end{cases} \quad (21)$$

$$U_2(x,0) = \Phi_2(x)\,,\ V_2(x,0) = \Psi_2(x)\ ,\ 0 < x \le +\infty \quad (22)$$

$$V_1(0,t) = \mu(t),\ t \ge 0. \quad (23)$$

Xarakteristikalar metididan foydalanib (18)-(23) masalayechimini qidiramiz.

Bunda har bir tenglamani mos xarakteristik chizig‘i ustida integrallab kerakli yechim topiladi. (18)-(23) sistema xarakteristikalari

$$\frac{dx}{dt} = \pm a_i$$

oddiy differensial tenglama orqali topiladi.

Xarakteristikalar metodidan foydalanib, (18)-(23) masala yechimini yozamiz

$U_1(x,t) = \Phi_1(x - a_1 t)e^{-b_1 t} + \int_0^t e^{-b_1(t-\eta)} F_1(x - a_1(t-\eta),\eta)d\eta,\quad -\infty < x \le 0,$ (24)

$V_1(x,t) = \Psi_1(x + a_1 t)e^{-b_1 t} + \int_0^t e^{-b_1(t-\eta)} G_1(x + a_1(t-\eta),\eta)d\eta,\ -\infty < x \le -a_1 t,$ (25) $V_1(x,t) = \mu\left(t + \frac{x}{a_1}\right) e^{\frac{b_1}{a_1}x} + \int_{t+\frac{x}{a_1}}^t e^{-b_1(t-\eta)} G_1(x + a_1(t-\eta),\eta)d\eta,$ $-a_1 t \le x \le 0,$ (26)

$U_2(x,t) = \mu\left(t - \frac{x}{a_2}\right) e^{-\frac{b_2}{a_2}x} + \int_{t-\frac{x}{a_2}}^t e^{-b_2(t-\eta)} F_2(x - a_2(t-\eta),\eta)d\eta,\ 0 \le x \le a_2 t,$ (27)

$U_2(x,t) = \Phi_2(x - a_2 t)e^{-b_2 t} + \int_0^t e^{-b_2(t-\eta)} F_2(x - a_2(t-\eta),\eta)d\eta,,\ a_2 t \le x \le \infty,$ (28)

$V_2(x,t) = \Psi_2(x + a_2 t)e^{-b_2 t} + \int_0^t e^{-b_2(t-\eta)} G_2(x + a_2(t-\eta),\eta)d\eta,\ 0 < x \le \infty.$ (29)

Agar yechimni ikkita deb teskarisidan faraz qilsak, bu yechimlar ayirmasi masala chiziqli ekanligidan yana yechim bo‘ladi hamda yechimning oshkor (aniq) ko‘rinishidan bu ayirma nolga tengligi kelib chiqadi, farazimiz ziddiyatga olib keladi. Demak, (18)-(23) masala yechimi yagona ekanligi kelib chiqadi. (15),(16),(17), (24)-(29) munosabatlar orqali (9)-(14) masala yechimi topiladi.

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