MITTAG – LIFFLER FUNKSIYASI VA UNI HISOBLASH USULLARI

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ANNOTATSIYA

Kasr tartibli xususiy hosilali differensial tenglamalarning fizik, biologik, mexanik, muhandislik va ko‘pgina boshqa sohalarda amaliy ahamiyatga ega ekanligi aniqlangandan keyin ularni o‘rganishga bo‘lgan qiziqish izlanuvchilar orasida keng tarqalib bormoqda. Masalan, kasr tartibli tenglamalar uchun manba yoki chegaraviy funksiyani topish masalalari zarralar tarqalishi (virus tarqalishi) jarayonlarini o‘rganishda muhim amaliy ahamiyatga ega.

Kalit so‘zlar: Kasr tartibli integrallar, matematik analiz, Riman integrali, grafiklar, kasr integro – differensial hisob, tolali polimerlar, deformatsiya.

ABSTRACT

After it was determined that differential equations with fractional order are of practical importance in physics, biology, mechanics, engineering and many other fields, the interest in studying them is spreading widely among researchers. For example, the problems of finding the source or boundary function for fractional equations are of great practical importance in the study of particle diffusion (virus diffusion) processes.

Keywords: fractional integrals, mathematical analysis, Riemann integral, graphs, fractional integro – differential calculus, fibrous polymers, deformation.

KIRISH

So‘nggi yillarda ko‘pgina hayotiy jarayonlarning matematik modelini tuzib, uni matematik usullar bilan yechish matematiklar ichida keng tarqaldi. Bu jarayonlar meditsina va texnikalar rivojlanishi bilan uzviy bog‘liqdir. Kasr tartibli integral va hosilalarning fizika, biologiya, meditsina va texnika sohalariga tadbiqi juda ko‘p bo‘lib, u bu sohalarning rivojlanishida muhim ahamiyatga ega. Shu sababli so‘nggi yillarda matematiklar orasida kasr tartibli hosila qatnashgan differensial va xususiy hosilali differensial tenglamalarni o‘rganishga bo‘lgan qiziqish ortib bormoqda.

Hozirgi kunda kasr tartibli xususiy hosilali differensial tenglamalarni va unga bog‘liq teskari masalalarni yechish bilan juda ko‘plab matematiklar shug'ullanishmoqda. Bu yo'nalishda olingan natijalarga misol qilib Sh.O. Alimov, R.R Ashurov, S.R. Umarov, M. Yamomota, Z. Li, A. Ashyaraliyev, B. Turmetov, Y. Zhang, H.T. Nguyn, A.V. Pskhu, A.S.Malik, E. Karimov va hokazolarning ishlarini aytish mumkin.

Mittag – Liffler funksiyasi va uni hisoblash usullari.

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)} \qquad (z\in\mathbb{C},\Re(\alpha)>0) \qquad (2.10)$$

(2.10) funksiya Mittag – Leffler funksiya deb ataladi.

Bu funksiya Mittag – Leffler tomonidan kiritilgan va uning nomi bilan nomlanadi. Bu funksiya kasr tartibli hosila va integrallarni hisoblashlarda foydalaniladi. Bu funksiyaning bir nechta xossalarini ko'rib chiqamiz.

Umumlashgan Mittag - Leffler funksiyalari. Umumlashgan Mittag – Leffler funksiyalari kasr tartibli hosilali chiziqli differensial tenglamalarini yechimini ifodalashda kerak bo'ladi.

Bir paramertli Mittag – Leffler funksiyasi quyidagi tenglik bilan aniqlanadi:

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}.$$

Ikki paramertli Mittag – Leffler funksiyasi quyidagi tenglik bilan aniqlanadi:

Aytaylik, $\alpha>0$ va β ixtiyoriy kompleks son bo'lsin. $E_{\alpha,\beta}(z)$ orqali biz ikkita parametrli Mittag - Leffler funksiyasini belgilaymiz

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \qquad (2.11)$$

Agar $\beta=1$ bolsa, biz bir parametrli Mittag-Leffler funksiyasiga ega bo'lamiz:

$$E_{\alpha,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}=E_\alpha(z).$$

Keyinchalik bizga yetarlicha katta manfiy argumentga ega Mittag-Leffler funksiyasining asimptotik bahosi kerak bo'ladi. Bu asimptotik bahosi quyidagi ko'rinishga ega.

$$|E_{\alpha,\beta}(-t)|\le\frac{C}{1+t}, \quad t>0, \qquad (2.12)$$

bu yerda β ixtiyoriy kompleks son. Bu baho asosan quyidagi asimptotik bahodan kelib chiqadi.

$$E_{\alpha,\beta}(-t)=\frac{t^{-1}}{\Gamma(\beta-\alpha)}+O(t^{-2}). \qquad (2.13)$$

Ikki parametrli Mittag-Leffler funksiyasi $E_{\alpha,\alpha}(-t)$ uchun (2.12) ga qaraganda yaxshiroq baho olish mumkin. Haqiqatan ham, quyidagi asimptotik bahodan foydalanib.

$$E_{\alpha,\alpha}(-t)=-\frac{t^{-2}}{\Gamma(-\alpha)}+O(t^{-3}), \qquad (2.14)$$

va bu $E_{\alpha,\alpha}(t)$ haqiqiy qiymatli analitik funksiya bo‘lsa, biz quyidagi tengsizlikni olishimiz mumkin:

$$|E_{\alpha,\alpha}(-t)|\le\frac{C}{1+t^2}, \quad t>0. \qquad (2.15)$$

Bundan tashqari, λ musbat son va $0<\varepsilon<1$ uchun quyidagi bahodan foydalanamiz:

$$|t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})|\le\frac{Ct^{\alpha-1}}{1+(\lambda t^{\alpha})^2}\le C\lambda^{\varepsilon-1}t^{\varepsilon\alpha-1}, \quad t>0, \qquad (2.16)$$

buni tekshirish oson. Haqiqatan ham, $t^{\alpha}\lambda<1$ bo‘lsa, u holda $t<\lambda^{-1/\alpha}$ va $t^{\alpha-1}=t^{\alpha-\varepsilon\rho}t^{\varepsilon\alpha-1}<\lambda^{\varepsilon-1}t^{\varepsilon\alpha-1}$.

Agar $t^{\alpha}\lambda\ge1$ bo‘lsa, u holda $\lambda^{-1}\le t^{\alpha}$ va $\lambda^{-2}t^{-\alpha-1}=\lambda^{-1+\varepsilon}\lambda^{-1-\varepsilon}t^{-\alpha-1}\le\lambda^{\varepsilon-1}t^{\varepsilon\alpha-1}$ bo‘ladi.

1 - tasdiq. Aytaylik, $0<\alpha<1$ bo‘lsin . U holda

$$E_{\alpha}(x)>0, \quad \frac{d}{dx}E_{\alpha}(x)>0, \quad x\in R. \qquad (2.17)$$

Isbot. $x\ge0$ bo‘lgani uchun (2.17) baho (1.12) ta’rifdan kelib chiqadi.

$x<0$ lar uchun Mittag-Leffler funksiyasining integral ko‘rinishidan foydalanamiz

$$E_{\alpha}(x)=\frac{\sin\alpha\pi}{\pi}\int_0^{\infty}\frac{e^{-t|x|^{1/\alpha}}}{1+2t^{\alpha}\cos\alpha\pi+t^{2\alpha}}t^{\alpha}dt>0.$$

U holda

$$\frac{d}{dx}E_{\alpha}(x)=|x|^{(1-\alpha)/\alpha}\cdot\frac{\sin\alpha\pi}{\alpha\pi}\int_0^{\infty}\frac{e^{-t|x|^{1/\alpha}}}{1+2t^{\rho}\cos\alpha\pi+t^{2\alpha}}t^{\alpha+1}dt>0.$$

1.1 - tasdiqdan foydalanib, (2.17) baho va $E_{\alpha}(0)=1$ tenglikka asosan biz quyidagiga ega bo‘lamiz.

1.2 - tasdiq. Argumentning manfiy qiymatlarida $E_{\alpha}(-x)$ Mittag-Leffler funksiyasi barcha $0<\alpha<1$ lar uchun monoton kamayuvchi funksiya va quyidagi tengsizliklar o‘rinli bo‘ladi:

$0 < E_{\alpha}(-x) < 1.$ (2.18)

1.3 - tasdiq. Aytaylik, $\alpha > 0$ va $\lambda > 0$ bo‘lsin. U holda barcha $t > 0$ lar uchun quyidagi tenglik o‘rinli bo‘ladi:

$$\int_0^t \eta^{\alpha-1} E_{\alpha,\alpha}(-\lambda\eta^{\alpha})d\eta = \frac{1}{\lambda}(1 - E_{\alpha}(-\lambda t^{\alpha})). \quad (2.19)$$

Isbot. Birinchidan, Mittag-Leffler funksiyasining hosilasini hisoblaymiz

$$\frac{d}{dt}E_{\alpha}(-\lambda t^{\alpha}) = -\alpha\lambda t^{\alpha-1}\sum_{n=1}^{\infty}\frac{n(-\lambda t^{\alpha})^{n-1}}{\Gamma(\alpha n+1)} = -\alpha\lambda t^{\alpha-1}\sum_{k=0}^{\infty}\frac{(k+1)(-\lambda t^{\alpha})^{k}}{\Gamma(\alpha(k+1)+1)}.$$

Agar $\Gamma(n+1) = n\Gamma(n)$ ekanligidan foydalansak,

$$\frac{d}{dt}E_{\rho}(-\lambda t^{\rho}) = -\lambda t^{\rho-1}\sum_{k=0}^{\infty}\frac{(-\lambda t^{\rho})^{k}}{\Gamma(\rho k+\rho)} = -\lambda t^{\rho-1}E_{\rho,\rho}(-\lambda t^{\rho})$$

tenglikni hosil qilamiz. E’tibor bering, bu yerda qator R da hadma - had differensiallanadi.

Quyidagi

$$\int_0^t \eta^{\alpha-1} E_{\alpha,\alpha}(-\lambda\eta^{\alpha})d\eta = -\frac{1}{\lambda}\int_0^t \frac{d}{d\eta}E_{\alpha}(-\lambda\eta^{\alpha})d\eta,$$

tenglikdan foydalanib, biz kerakli natijaga erishamiz.

4 - tasdiq. Aytaylik $0 < \alpha < 1$ va $\lambda > 0$ bo‘lsin. U holda

$$\frac{d}{dt}[t^{\alpha}E_{\alpha,\alpha+1}(-\lambda t^{\alpha})] > 0, \quad t > 0,$$

ya’ni $t^{\alpha}E_{\alpha,\alpha+1}(-\lambda t^{\alpha})$ funksiya $t > 0$ da qat’iy o‘suvchi bo‘ladi.

Isbot. (2.11) va hadma-had integrallashdan foydalanib, quyidagini hosil qilamiz:

$$\int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda\eta^{\rho})d\eta = t^{\rho}E_{\rho,\rho+1}(-\lambda t^{\rho}), \quad (2.20)$$

yoki 3 – tasdiqqa ko‘ra,

$$t^{\alpha}E_{\alpha,\alpha+1}(-\lambda t^{\alpha}) = \frac{1}{\lambda}(1 - E_{\alpha}(-\lambda t^{\alpha}))$$

bo‘ladi. 2 - tasdiqni qo‘llab aytilgan natijani olamiz.

5 - tasdiq. Aytaylik, $0 < \alpha < 1$ va $\lambda > 0$ bo‘lsin. U holda barcha musbat t lar uchun

$$J_t^{\alpha-1}(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})) = E_{\alpha}(-\lambda t^{\alpha}) \quad (2.21)$$

tenglik o‘rinli bo‘ladi.

Isbot. Kasr tartibli integralning ta’rifidan biz quyidagiga ega bo‘lamiz:

$$J_t^{\alpha-1}(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha}))=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\xi^{\alpha-1}E_{\alpha,\alpha}(-\lambda\xi^{\alpha})}{(t-\xi)^{\alpha}}d\xi=$$

$$=\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{\infty}\frac{(-\lambda)^j}{\Gamma(\alpha j+\alpha)}\int_0^t\frac{\xi^{\alpha-1+\alpha j}}{(t-\xi)^{\alpha}}d\xi=\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{\infty}\frac{(-\lambda)^j t^{\alpha j}}{\Gamma(\alpha j+\alpha)}\int_0^1 s^{\alpha-1+\alpha j}(1-s)^{-\alpha}ds.$$

Boshqa tomondan, Eylerning $B(a,b)$ beta funksiyasining ta’rifi va (2.9) xossasidan

$$B(\alpha+\alpha j,1-\alpha)=\int_0^1 s^{\alpha-1+\alpha j}(1-s)^{-\alpha}ds=\frac{\Gamma(\alpha+\alpha j)\Gamma(1-\alpha)}{\Gamma(\alpha j+1)}.$$

Oxirgi tenglikka integralning o‘rniga qo‘yib va $E_{\alpha}(z)$ Mittag-Leffler funksiyasining ta’rifidan tasdiqning isboti kelib chiqadi.

Ba’zi ikki paramertli Mittag – Leffler funksiyalarini hisoblashda olingan natijalar:

Masalan:

$$E_{1,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)}=\sum_{k=0}^{\infty}\frac{z^k}{k!}=e^z,$$

$$E_{1,2}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+2)}=\frac{1}{z}\sum_{k=0}^{\infty}\frac{z^{k+1}}{(k+1)!}=\frac{1}{z}\sum_{k=0}^{\infty}\frac{z^{k+1}}{(k+1)!}=$$

$$=\frac{1}{z}\sum_{n=1}^{\infty}\frac{z^n}{n!}=\frac{1}{z}\sum_{n=0}^{\infty}\frac{z^n}{n!}-\frac{1}{z}=\frac{1}{z}e^z-\frac{1}{z}=\frac{e^z-1}{z},$$

$$E_{1,3}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+3)}=\frac{1}{z^2}\sum_{k=0}^{\infty}\frac{z^{k+2}}{(k+2)!}=\frac{e^z-1-z}{z^2},$$

$$E_{\alpha,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}=E_{\alpha}(z)$$

1) $E_{\frac{1}{2},\frac{1}{2}}(\mathrm{t})\Rightarrow$ integralni hisoblang.

$$\int_0^t E_{\frac{1}{2},\frac{1}{2}}(\xi)d\xi=\sum_{k=0}^{\infty}\int_0^t\frac{\xi^k}{\Gamma(\frac{1}{2}k+\frac{1}{2})}d\xi=$$

$$=\sum_{k=0}^{\infty}\frac{1}{\Gamma(\frac{1}{2}k+\frac{1}{2})}\int_0^t\xi^k d\xi=\sum_{k=0}^{\infty}\frac{1}{\Gamma(\frac{1}{2}k+\frac{1}{2})}\frac{\xi^k}{k+1}\Big|_0^t=$$

$$=\sum_{k=0}^{\infty}\frac{1}{\Gamma(\frac{1}{2}k+\frac{1}{2})}\left[\frac{t^{k+1}}{k+1}\right]=\sum_{k=0}^{\infty}\frac{t^k\cdot t}{\Gamma(\frac{1}{2}k+\frac{1}{2})(k+1)}=$$

$$=\frac{t}{2}\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\frac{1}{2}k+\frac{3}{2})}=\frac{t}{2}E_{\frac{1}{2},\frac{3}{2}}(\mathrm{t})$$

2) $\int_0^t \xi^{\mu-1}\cdot E_{\alpha,\mu}(\xi^{\alpha})\,\mathrm{d}\xi \Rightarrow$ integralni hisoblang.

$$\int_0^t \xi^{\mu-1}\cdot E_{\alpha,\mu}(\xi^{\alpha})\,\mathrm{d}\xi=\int_0^t \xi^{\mu-1}\cdot\sum_{k=0}^{\infty}\frac{\xi^{\alpha k}}{\Gamma(\alpha k+\mu)}\mathrm{d}\xi=$$

$$\sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+\mu)}\int_o^t \xi^{\alpha k+\mu-1}d\xi=\sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+\mu)})(\frac{\xi^{\alpha k+\mu}}{\alpha k+\mu})\Big|_0^t d\xi=$$

$$=\sum_{k=0}^{\infty}\frac{t^{\alpha k+\mu}}{(\alpha k+\mu)\Gamma(\alpha k+\mu)}=t^{\mu}\sum_{k=0}^{\infty}\frac{(t^{\alpha})^k}{\Gamma(\alpha k+\mu+1)}=t^{\mu}E_{\alpha,\mu+1}(\mathrm{t}^{\alpha})$$

$$\int_0^t \xi^{\mu-1}\cdot E_{\alpha,\mu}(\xi^{\alpha})\,\mathrm{d}\xi=t^{\mu}E_{\alpha,\mu+1}(\mathrm{t}^{\alpha}) \qquad (2.12)$$

(2.12) formuladan xususiy hosilali differensial tenglamalarni yechishda foydalaniladi.

Masalan: $\int_0^t \xi^{\frac{1}{2}}\cdot E_{\frac{1}{2},\frac{3}{2}}(\xi^{\frac{1}{2}})\,\mathrm{d}\xi \Rightarrow$ integralni hisoblang.

Yuqoridagi (2.12) formuladan foydalanamiz:

$$\int_0^t \xi^{\frac{1}{2}}\cdot E_{\frac{1}{2},\frac{3}{2}}(\xi^{\frac{1}{2}})\,\mathrm{d}\xi=t^{\frac{3}{2}}E_{\frac{1}{2},\frac{5}{2}}(\xi^{\frac{1}{2}}).$$

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