CAPUTA MA'NOSIDA KASR TARTIBLI HOSILALAR VA UNI HISOBLASH USULLARI

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ANNOTATSIYA

Maqolada kasr tartibli hisoblash rivojlanishining qisqacha tarixiy sharhi berilgan, butun sonli bo'lmagan hosilalar bilan ishlash uchun matematik tahlilning maxsus funksiyalari qaralgan. Caputo va Riman – Liovilning kasr tartibli hosilalari qaralgan.

Kalit so'zlar: Caputo kasr tartibli hosilasi, Kasr tartibli integrallar, matematik analiz, Riman integrali, grafiklar, kasr integro – differensial hisob, tolali polimerlar, deformatsiya.

ABSTRACT

The paper provides a brief historical overview of the development of fractional arithmetic, considering the special functions of mathematical analysis for working with non- integer products. Fractional – order derivatives of Caputo and Riemann – Lioville are considered.

Keywords: Caputo fractional product, fractional integrals, mathematical analysis, Riemann integral, graphs, fractional integro – differential calculus, fibrous polymers, deformation.

Caputo ma'nosida kasr tartibli hosilalar va uni hisoblash usullari.

Matematik fizikaning juda ko'p masalalarini xususiy hosilali tenglamalar ko'rinishida tavsiflash mumkin. Bunday tavsivlashda unga mos bo'lgan chegaraviy masalalar yechimini tabiiy aniqlash imkoniyati mavjud va shu bilan birga, ularni yechishga ma'lum usullarni qo'llash mumkin. Tabiatda uchraydigan jarayonlarni kasr tartibli tenglamalar aniqroq ifoda etadi. Hozirga kelib ularni amaliy masalalarga tadbig'iga talab ortganligi tufayli izlanishlar kuchaydi va masalalarning qo'yilishiga aniqlik kiritilib, mutahassislar oldiga butunlay yangi tipdagi masalalar qo'yildi.

Riman-Luivill hosilasiga qo'shimcha tarzda 1967 yil Caputo tomonidan kiritilgan kasr tartibli differensialning ta'rifini va ba'zi xossalarini keltiramiz.

Aytaylik, $[a,b]$ segment R dagi biror chegaralangan oraliq va $\alpha \in C(\Re(\alpha) \geq 0)$ bo'lib,

$D_{a+}^{\alpha}\left[y(t)\right](x) \equiv (D_{a+}^{\alpha}y)(x)$ va $\mathrm{D}_{b-}^{\alpha}\left[y(t)\right](x) \equiv (D_{b-}^{\alpha}y)(x)$ lar Riman-Luivill ma'nosida α - kasr tartibli hosilalar bo'lsin.

$({}^{C}D_{a+}^{\alpha}y)(x)$ va $({}^{C}D_{b-}^{\alpha}y)(x)$ Caputo ma'nosida $\alpha \in C$ $(\Re(\alpha) \geq 0)$ kasr tartibli hosilalar Riman - Luivill ma'nosida α - kasr tartibli hosilalar yordamida quyidagicha aniqlanadi:

$$({}^{C}D_{a+}^{\alpha}y)(x) := \left(D_{a+}^{\alpha}\left[y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(t-a)^{k} \right] \right)(x), \qquad (1.1)$$

$$({}^{C}D_{b-}^{\alpha}y)(x) := \left(D_{b-}^{\alpha}\left[y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(b)}{k!}(b-t)^{k} \right] \right)(x), \qquad (1.2)$$

bu yerda, agar $\alpha \notin N$ bo'lsa, $n = [\Re(\alpha)] + 1$, agar α natural son bo'lsa, $n = \alpha$ bo'ladi.

Ushbu (1.1) va (1.2) larga mos ravishda Caputo ma'nosida chap va o'ng hosilalar deb ataladi.

Agar $0 < \Re(\alpha) < 1$ bo'lsa, (1.1) va (1.2) larga mos ravishda Caputo ma'nosida chap va o'ng hosilalar quyidagicha aniqlanadi:

$$({}^{C}\mathrm{D}_{a+}^{\alpha}y)(x) = \left(D_{a+}^{\alpha}\left[y(t) - y(a) \right] \right)(x), \qquad (1.3)$$

$$({}^{C}\mathrm{D}_{b-}^{\alpha}y)(x) = \left(D_{b-}^{\alpha}\left[y(t) - y(b) \right] \right)(x). \qquad (1.4)$$

Agar $\alpha \notin N$ bo'lib, $y(x)$ funksiyaning $({}^{C}D_{a+}^{\alpha}y)(x)$ va $({}^{C}D_{b-}^{\alpha}y)(x)$ Caputo ma'nosida kasr tartibli hosilalar va $(D_{a+}^{\alpha}y)(x)$ va $(D_{b-}^{\alpha}y)(x)$ Riman - Luivill ma'nosida hosilalar mavjud bo'lsa, u holda quyidagi tengliklar o'rinli bo'ladi:

$$({}^{C}D_{a+}^{\alpha}y)(x) = (D_{a+}^{\alpha}y)(x) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha}, \qquad (1.5)$$

$$({}^{C}\mathrm{D}_{b-}^{\alpha}y)(x) = (\mathrm{D}_{b-}^{\alpha}y)(x) - \sum_{k=0}^{n-1} \frac{y^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-x)^{k-\alpha}, \qquad (1.6)$$

bu yerda $(n = [\Re(\alpha)] + 1)$.

Agar $0<\Re(\alpha)<1$ bo‘lsa, u holda quyidagicha bo‘ladi

$$({}^C D_{a+}^{\alpha} y)(x) = (D_{a+}^{\alpha} y)(x) - \frac{y(a)}{\Gamma(1-\alpha)}(x-a)^{-\alpha}, \quad (1.7)$$

$$({}^C D_{b-}^{\alpha} y)(x) = (D_{b-}^{\alpha} y)(x) - \frac{y(b)}{\Gamma(1-\alpha)}(b-x)^{-\alpha}. \quad (1.8)$$

Agar $\alpha \notin N$ bo‘lib, $y(a) = y'(a) = \cdots = y^{(n-1)}(a) = 0,$ bunda $(n = [\Re(\alpha)] + 1)$ bo‘lsa, u holda Caputo va Riman - Luivill, ya’ni

$$({}^C D_{a+}^{\alpha} y)(x) = (D_{a+}^{\alpha} y)(x) \quad (1.9)$$

bo‘ladi.

Xuddi shunga o‘xshash $y(b) = y'(b) = \cdots = y^{(n-1)}(b) = 0$ bo‘lsa, u holda

$$({}^C D_{b-}^{\alpha} y)(x) = (D_{b-}^{\alpha} y)(x) \quad (1.10)$$

bo‘ladi. Xususan, $0<\Re(\alpha)<1$ bo‘lib, $y(a)=0$ bo‘lsa,

$$({}^C D_{a+}^{\alpha} y)(x) = (D_{a+}^{\alpha} y)(x) \quad (1.11)$$

bo‘ladi. Xuddi shunga o‘xshash $y(b)=0$ bo‘lsa, u holda

$$({}^C D_{b-}^{\alpha} y)(x) = (D_{b-}^{\alpha} y)(x) \quad (1.12)$$

bo‘ladi.

Agar $\alpha \in N$ bo‘lsa, u holda Caputo ma’nosidagi hosilalar oddiy hosilalar bilan ustma-ust tushadi, ya’ni:

$$({}^C D_{a+}^{n} y)(x) = y^{(n)}(x) \quad (1.13)$$

$$({}^C D_{b-}^{n} y)(x) = (-1)^n y^{(n)}(x) \quad (1.14)$$

bo‘ladi. Biror funksiyaning Caputo ma’nosida hosilalari mavjud bo‘lishligi uchun shu funksiyaning Riman - Luivill ma’nosida kasr tartibli hosilalar mavjud bo‘lishi kerak. Buning uchun $y(x) \in AC^n[a,b]$ bo‘lishi kerak, ya’ni $y(x)$ funksiya absalyut funksiya bo‘lishi kerak.

1.1-teorema. $\Re(\alpha) \ge 0$ va $n = [\Re(\alpha)] + 1$ bo‘lsin. Agar $y(x) \in AC^n[a,b]$ bo‘lsa, u holda $[a,b]$ da $({}^C D_{a+}^{\alpha} y)(x)$ va $({}^C D_{b-}^{\alpha} y)(x)$ Caputo ma’nosida kasr tartibli hosilalar mavjud,

a) agar $\alpha \notin N$ bo‘lsa, $({}^C D_{a+}^{\alpha} y)(x)$ va $({}^C D_{b-}^{\alpha} y)(x)$ lar quyidagicha ko‘rinishda bo‘ladi:

$$({}^C D_{a+}^{\alpha} y)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y^{(n)}(t)dt}{(x-t)^{\alpha-n+1}} =: (I_{a+}^{n-\alpha} D^n y)(x) \quad (1.15)$$

$$({}^{C}D_{b-}^{\alpha}y)(x)=\frac{(-1)^{n}}{\Gamma(1-\alpha)}\int_{x}^{b}\frac{y^{(n)}(t)dt}{(t-x)^{\alpha-n+1}}=:(-1)^{n}(I_{b-}^{n-\alpha}D^{n}y)(x) \qquad (1.16)$$

bunda, $\mathrm{D}=\dfrac{d}{dx}$.

Xususan, $0<\Re(\alpha)<1$ va $y(x)\in AC[a,b]$ bo‘lsa,

$$({}^{C}D_{a+}^{\alpha}y)(x)=\frac{1}{\Gamma(1-\alpha)}\int_{a}^{x}\frac{y'(t)dt}{(x-t)^{\alpha}}=(\mathrm{I}_{a+}^{1-\alpha}Dy)(x), \qquad (1.17)$$

$$({}^{C}D_{b-}^{\alpha}y)(x)=-\frac{1}{\Gamma(1-\alpha)}\int_{x}^{b}\frac{y'(t)dt}{(t-x)^{\alpha}}=:-(I_{b-}^{1-\alpha}Dy)(x) \qquad (1.18)$$

bo‘ladi.

a) agar $\alpha\in N$ bo‘lsa, $({}^{C}D_{a+}^{\alpha}y)(x)$ va $({}^{C}D_{b-}^{\alpha}y)(x)$ lar mos ravishda (1.13) va (1.14) formulalar bilan aniqlanadi. Xususan,

$$({}^{C}D_{a+}^{0}y)(x)=({}^{C}D_{b-}^{0}y)(x)=y(x) \qquad (1.19)$$

bo‘ladi.

Xuddi yuqoridagi teoremaga o‘xshash agar $y(x)$ funksiya n tartibli hosilalari bilan uzluksiz funksiyalar sinfiga tegishli bo‘lsa, u holda quyidagi teorema o‘rinli bo‘ladi.

1.2-teorema. $\Re(\alpha)\geq 0$ va $n=[\Re(\alpha)]+1$ bo‘lsin. Agar $y(x)\in \mathrm{C}^{n}[a,b]$ bo‘lsa, u holda $[a,b]$ da $({}^{C}D_{a+}^{\alpha}y)(x)$ va $({}^{C}D_{b-}^{\alpha}y)(x)$ Caputo ma’nosida kasr tartibli hosilalar mavjud,

a) agar $\alpha\notin N$ bo‘lsa, $({}^{C}D_{a+}^{\alpha}y)(x)$ va $({}^{C}D_{b-}^{\alpha}y)(x)$ lar (1.15) va (1.16) formulalar bilan aniqlanadi. Bundan tashqari quyidagi tengliklar o‘rinli bo‘ladi:

$$({}^{C}D_{a+}^{\alpha}y)(a)=({}^{C}D_{b-}^{\alpha}y)(\mathrm{b})=0. \qquad (1.20)$$

Xususan, $0<\Re(\alpha)<1$ bo‘lsa, mos ravishda (1.17) va (1.18) formulalar bilan aniqlanadi.

b) agar $\alpha\in N$ bo‘lsa, $({}^{C}D_{a+}^{\alpha}y)(x)$ va $({}^{C}D_{b-}^{\alpha}y)(x)$ lar mos ravishda (1.13) va (1.14) formulalar bilan aniqlanadi. Xususan,

$$({}^{C}D_{a+}^{0}y)(x)=({}^{C}D_{b-}^{0}y)(x)=y(x) \qquad (1.21)$$

bo‘ladi.

1.1-natija. $\Re(\alpha)\geq 0$ va $n=[\Re(\alpha)]+1$ bo‘lsin.

a) agar $y(x)\in C^n[a,b]$ bo‘lsa, u holda $({}^C D^{\alpha}_{a+}y)(x)$ va $({}^C D^{\alpha}_{b-}y)(x)$ Caputo ma’nosida kasr tartibli hosilalar $C[a,b]$ sinfga tegishli bo‘ladi. Bundan tashqari quyidagi baholashlar o‘rinli bo‘ladi:

$$\left\|{}^c D^{\alpha}_{\alpha+}y\right\|_{C_\alpha} \le k_\alpha \|y\|_{C^n} \quad \text{va} \quad \left\|{}^c D^{\alpha}_{b+}y\right\|_{C_b} \le k_\alpha \|y\|_{C^n}, \tag{1.22}$$

bu yerda

$$k_\alpha = \frac{(b-a)^{n-\Re(\alpha)}}{|\Gamma(n-\alpha)|[n-\Re(\alpha)+1]}. \tag{1.23}$$

b) agar $\alpha\in N$ bo‘lsa, $({}^C D^{\alpha}_{a+}y)(x)$ va $({}^C D^{\alpha}_{b-}y)(x)$ lar $C[a,b]$ sinfga tegishli bo‘ladi. Bundan tashqari quyidagi tengliklar o‘rinli bo‘ladi:

$$\left\|{}^c D^{\alpha}_{\alpha+}y\right\|_{C_\alpha} \le k_\alpha \|y\|_{C^n} \quad \text{va} \quad \left\|{}^c D^{\alpha}_{b+}y\right\|_{C_b} \le k_\alpha \|y\|_{C^n} \tag{1.24}$$

1) $f(t)=1$ $0\prec\alpha\prec1$ Caputo ma’nosida kasr tartibli hosilasini hisoblang.

$$\begin{gathered}{}^c D^{\alpha}[1] = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\xi)^{1-\alpha-1}(1)'\mathrm{d}\xi = \frac{1}{\Gamma(1-\alpha)}\int_0^t 0 d\xi = \\ = \frac{1}{\Gamma(1-\alpha)}C\Big|_0^t = \frac{1}{\Gamma(1-\alpha)}(C-C)=0\end{gathered} \tag{1.25}$$

(1.25) kelib chiqadiki o‘zgarmast sonning $0\prec\alpha\prec1$
oraliqdagi Caputo hosilasi nolga teng.

1) $f(t)=t^{\mu}$ $0\prec\alpha\prec1$ Caputo ma’nosida kasr tartibli hosilasini hisoblang.

$${}^c D^{\alpha} f(t) = I^{1-\alpha}\frac{d}{dt}(t^{\mu}) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\xi)^{1-\alpha-1}(\xi^{\mu})'d\xi =$$

$$= \frac{1}{\Gamma(1-\alpha)}\int_0^t \mu(t-\xi)^{-\alpha}\xi^{\mu-1}d\xi\begin{bmatrix}\xi = ty\\ d\xi = tdy\end{bmatrix} =$$

$$= \frac{1}{\Gamma(1-\alpha)}\int_0^1 \mu t^{\mu-\alpha}(1-y)^{-\alpha}y^{\mu-1}dy = \frac{\mu t^{\mu-\alpha}}{\Gamma(1-\alpha)}B(\mu,1-\alpha) =$$

$$= \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\alpha)}t^{\mu-\alpha} \tag{1.26}$$

1) $f(t)=e^t$ $0\prec\alpha\prec1$ Caputo ma’nosida kasr tartibli hosilasini hisoblang.

Yuqoridagi Caputo hosilasiga o‘xshab hisoblasak quyida formula kelib chiqadi.

$${}^c D^{\alpha}(e^t) = t^{1-\alpha}E_{1,2-\alpha}(\mathrm{t}) \tag{1.27}$$

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