TRIGONOMETRIK MASALALARNING GEOMETRIK YECHIMLARI

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ANNOTATSIYA

Mazkur maqolada ko‘p uchraydigan triganametrik ifodalarni faqat algebrayik usulda emas baliki geometrik shakillar yordamida ham ishlash mumkinligini ko‘rib chiqmiz mavzusiga oid ba’zi masalalarning yechimlari keltirilgan. Bular orqali o‘quvchi masalalarini bir xil usulda emas balki, boshqacha kreativ fikrlash orqali ham yechishi mumkinligini o‘rganadi.

Kalit so‘zlar: Uchburchak, tengsizlik, isbot, kosinuslar teoremas, perimetri, bissektrisa, burchak.

Ushbu maqola olimpiadada ishtirok etish va g‘olib bo‘lish istagidagi iqtidorli talabalar uchun yaratilgan. Maqola mustaqil o‘rganuvchilar uchun qulay bo‘lib, undagi ko‘pgina masalalarning yechimlari bilan berilgan.

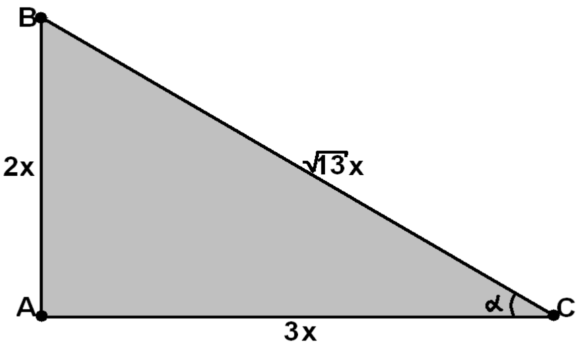
Ushbu mavzu masalalarini yechishning asosiy g‘oyasi qiymatlari masala shartiga mos keluvchi burchaklarni o‘z ichiga oladigan uchburchaklar bilan ishlashdir. Ushbu masalalarni yechish uchun Pifagor, sinuslar, kosinuslar teoremalari va uchburchaklarning turli xossalari qo‘llaniladi. Boshqa yondashuv birlik radiusli doira va burchaklarni hosil qiluchi birlik vektorlar bilan ishlash qobiliyatini talab qiladi. Masalalarni shu tarzda yechish uchun Ptolemey teoremasi va muntazam ko‘pyoqlarning xossalari bilan tanishish kerak bo‘ladi.

Masala 1-Ifodaning qiymatini toping: $2\sqrt{13}\cos\left(arctg\frac{2}{3}\right)$

Yechim. $a = arctg\frac{2}{3}, a$ О Ў Ю a o‘tkir burchak bo‘lsin.

a - burchak tangensi $\frac{2}{3}$ ga teng.

Tomonlari 2 va 3 ga teng bo‘lgan uchburchak yasaymiz. (2.2.1-rasm.)



1-rasm.

Pifagor teoremasi bo‘yicha: $BC = \sqrt{AB^2 + BC^2} = \sqrt{4x^2 + 9x^2} = \sqrt{13}x$

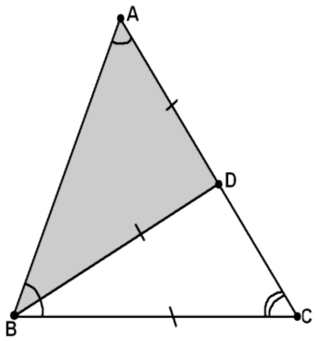
Bundan, $\cos a = \frac{AC}{BC} = \frac{3x}{\sqrt{13}x} = \frac{3}{\sqrt{13}}$, $2\sqrt{13}\cos\left(arctg\frac{2}{3}\right) = 2\sqrt{13}\cdot\frac{3}{\sqrt{13}} = 6$

Javob: 6.

Masala 2

$\cos\frac{p}{5} - \cos\frac{2p}{5} = \frac{1}{2}$ tengligini isbotlang.

Birinchi yechim. Isbotlash uchun $\frac{p}{5}, \frac{2p}{5}, \frac{2p}{5}$ burchakli ABC teng yonli uchburchagini ko‘rib chiqamiz (2-rasm.) va B burchakdan BD bissektrisasini chizamiz.



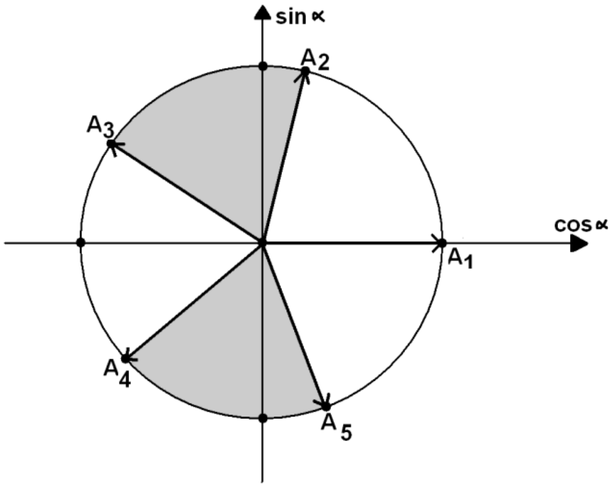
2-rasm.

U holda $BC = BD = AD$ bo‘ladi. Aniqlik uchun $BC = 1$ bo‘lsin. ABD va BCD uchburchaklardan $AB = 2\cos\frac{p}{5}, CD = 2\cos\frac{2p}{5}$ ega bo‘lamiz va bu tengliklardan $AD = AB - AC$ munosabat kelib chiqadi.

Ikkinchi yechim.

$\overrightarrow{OA_1}(1;0)$,

$\overrightarrow{OA_2}(\cos\frac{2p}{5}; \sin\frac{2p}{5})$



$$\vec{OA_3}\left(\cos\frac{4p}{5};\sin\frac{4p}{5}\right)$$

$$\vec{OA_4}\left(\cos\frac{6p}{5};\sin\frac{6p}{5}\right)$$

$$\vec{OA_5}\left(\cos\frac{8p}{5};\sin\frac{8p}{5}\right)$$

2(1)-rasm.

Vektorlar yig'indisini qarasak $\vec{OA_1}+\vec{OA_2}+\vec{OA_3}+\vec{OA_4}+\vec{OA_5}=0$ bundan

$$\cos\frac{2p}{5}+\cos\frac{4p}{5}+\cos\frac{6p}{5}+\cos\frac{8p}{5}+1=0$$

$$\cos\frac{2p}{5}-\cos\frac{p}{5}-\cos\frac{p}{5}+\cos\frac{2p}{5}+1=0$$

$$2\cos\frac{2p}{5}-2\cos\frac{p}{5}=-1$$

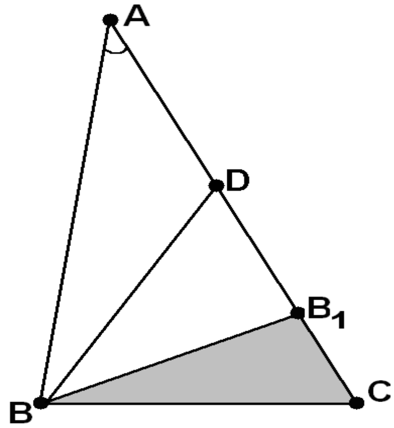
$$\cos\frac{p}{5}-\cos\frac{2p}{5}=\frac{1}{2}$$

$\cos\frac{p}{5}-\cos\frac{2p}{5}=\frac{1}{2}$ tenglikni to'g'riligi ko'rsatildi.

Masala 3 $\dfrac{1}{\sin\frac{p}{7}}=\dfrac{1}{\sin\frac{2p}{7}}+\dfrac{1}{\sin\frac{3p}{7}}$ tenglikni isbotlang.

Yechish: 1-hol.

Bizga faqat yuqorida $\frac{p}{7}$ burchakli va $BB_1 = 1$ balandlikli teng yonli ABC uchburchakni ko‘rib chiqish kifoya (2.2.3-rasm.).



3-rasm.

ABB_1, BDB_1, BCB_1 uchburchaklardan $AB = AC = \frac{1}{\sin a}$ $BD = AD = \frac{1}{\sin\frac{2p}{7}}$, $BC = \frac{1}{\sin\frac{3p}{7}}$ topib olamiz.

Bundan $AC = AD + CD$ kelib chiqadi.

Yechish: 2-hol.

Umumiy maxrajga keltiramiz:

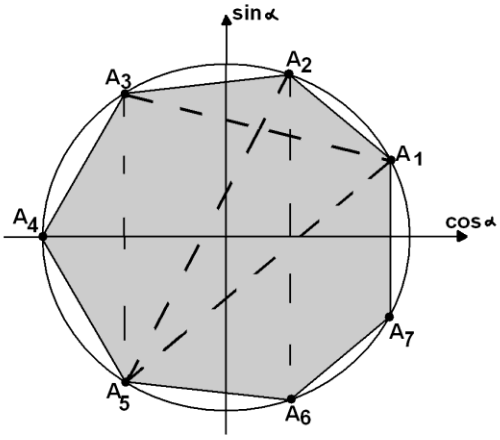
$$\sin\frac{2p}{7}\sin\frac{3p}{7} = \sin\frac{p}{7}\sin\frac{3p}{7} + \sin\frac{p}{7}\sin\frac{2p}{7} (*)$$

Birlik radiusli aylanaga ichki chizilgan $A_1A_2A_3A_4A_5$ ni ko‘rib chiqamiz. (3(1)-rasm.) Ushbu ko‘pburchakka Ptolemey teoremasini qo‘llaymiz (ichki chizilgan to‘rtburchakda diagonallar ko‘paytmasi qarama- qarshi tomonlarning ko‘paytmalarining yig‘indisiga teng).

$$A_1A_3 \text{Ч} A_2A_5 = A_3A_5 \text{Ч} A_1A_2 + A_2A_3 \text{Ч} A_1A_5 \ (1)$$

$A_2A_5 = A_1A_5 = A_2A_6$, $A_3A_5 = A_1A_3$, $A_1A_2 = A_1A_7 = A_2A_3$ (1)-tenglamaga qo‘yib quyidagi tenglikga ega bo‘lamiz:

$$A_3A_5 \text{Ч} A_2A_6 = A_3A_5 \text{Ч} A_1A_7 + A_1A_7 \text{Ч} A_2A_6 \ (2)$$



3(1)-rasm.

A_2A_6, A_3A_5, A_1A_7 kesmalarning Oy o‘qidagi proyeksiyalarini topib va $A_1\left(\cos\frac{\pi}{7};\sin\frac{\pi}{7}\right)$, $A_2\left(\cos\frac{3\pi}{7};\sin\frac{3\pi}{7}\right)$, $A_3\left(\cos\frac{2\pi}{7};\sin\frac{2\pi}{7}\right)$ nuqtalarning koordinatalarini hisobga olib quyidagi tengliklarga ega bo‘lamiz:

$A_2A_6 = 2\sin\frac{3\pi}{7}$, $A_3A_5 = 2\sin\frac{2\pi}{7}$, $A_1A_7 = 2\sin\frac{\pi}{7}$ olingan natijalarni ikkinchi tenglikga qo‘yamiz (2):

$$2\sin\frac{2\pi}{7}\cdot 2\sin\frac{3\pi}{7} = 2\sin\frac{2\pi}{7}\cdot 2\sin\frac{\pi}{7} + 2\sin\frac{\pi}{7}\cdot 2\sin\frac{3\pi}{7} \; : 4$$

$$\sin\frac{2\pi}{7}\cdot\sin\frac{3\pi}{7} = \sin\frac{2\pi}{7}\cdot\sin\frac{\pi}{7} + \sin\frac{\pi}{7}\cdot\sin\frac{3\pi}{7}$$

Tenglikga ega bo‘ldik $(*)$. $\frac{1}{\sin\frac{\pi}{7}} = \frac{1}{\sin\frac{2\pi}{7}} + \frac{1}{\sin\frac{3\pi}{7}}$ tenglikni to‘g‘riligi ko‘rsatildi.

Masala 4

$tg\frac{3\pi}{7} - 4\sin\frac{\pi}{4} = \sqrt{7}$ tengligini isbotlang.

Yechish: Biz foydalanadigan asosiy ayniyatlar:

$$2\cos a\cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a\sin b = \cos(a-b) - \cos(a+b)$$

(1)

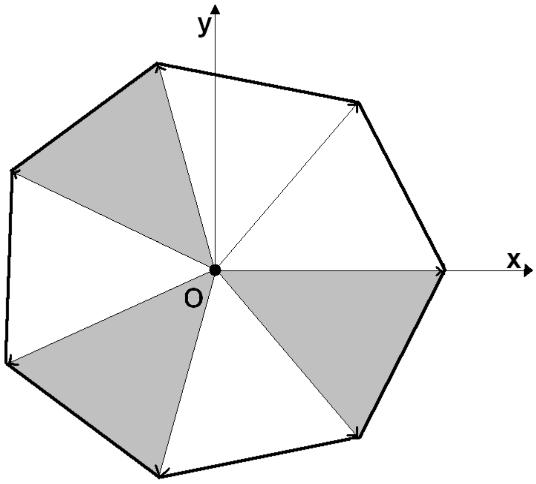
$$2\sin a\cos b = \sin(a+b) + \cos(a-b)$$

va ularning xususiy hollari ($a = b$ bo‘lganda).

Quyidagicha belgilash kiritamiz $h = \frac{p}{7}$. Quyidagilardan foydalanish bizga qulay bo‘ladi, $\sin 3h = \sin 4h, \sin h = \sin 6h,$ $\cos 2h = - \cos 5h,$ $\cos 6h = \cos 8h = - \cos h,$ va h. Biz ushbu belgilashlarni alohida qayd etamiz. Asosiy bu burchaklarning xossalari: $2\cos h + 2\cos 4h + 2\cos 6h = - 1,$ (2)

yoki unga ekvivalent, $2\cos h + 2\cos 3h + 2\cos 5h = 1,$ (2ў)

Bu yerda tenglikning geometrik isboti (2). $(1;0)$ nuqtadan chiqadigan vektorlarni ko‘rib chiqamiz (19-rasm.). Quyidagi rasmdagi vektorlarning yig‘indisi 0 ga teng bo‘lgani uchun (axir, markaz atrofida $2h$ burchakka burilganda, u o‘zgarishi mumkin emas!), ularning Ox o‘qidagi proyeksiyalari yig‘indisi ham 0 ga eng.



4-rasm.

(boshqa isbot (2) yoki (2ў) ni ikkala qismni $\sin h$ ga ko‘paytirish va (1) ko‘paytmani yig‘indiga aylantirish orqali olish mumkin.) Shunday qilib, keyingi tenglikni isbotlashimiz kerak $tg3h - 4\sin h = \sqrt{7}$ (3) ikkala qismini $\cos 3h$ ga ko‘paytirib, chap tomonni quyidagicha o‘zgartiramiz:

$L = \sin 3h - (2\sin 4h - \sin 2h) = 2\sin 2h - \sin 4h$ (4)

Bu sonning kvadrati (musbat!) isbotlash kerak $7\cos^2 3h$ va haqiqatan ham,

$$7\cos^2 3h - L^2 = 7\frac{1 + \cos 6h}{2} - (2 - 2\cos 4h) + 2(\cos 2h - 2\cos 6h) - \frac{1 - \cos 6h}{2} =$$

$$2\cos 2h + 2\cos 4h + 2\cos 6h + 1 = 0$$

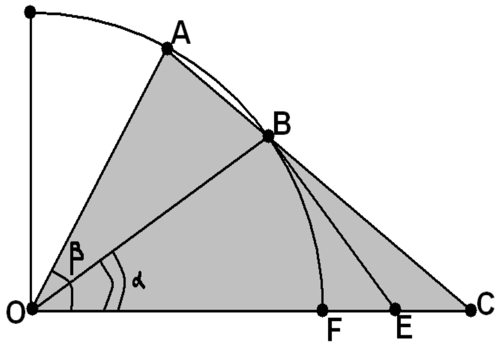
Tenglikni to‘g‘riligi ko‘rsatildi.

Masala 5 $0 < a < b$ J $\frac{p}{2}$ uchun $\frac{\sin a}{a} > \frac{\sin b}{b}$ ekanligini isbotlang.

Yechish:

OCB va OCA uchburchaklardan (5-rasm.) sinuslar teoremasiga ko‘ra: $\dfrac{AC}{\sin b} = \dfrac{OA}{\sin C}$,

$\dfrac{BC}{\sin a} = \dfrac{OB}{\sin C = \dfrac{OA}{\sin C}}$ Ю $\dfrac{AC}{\sin b} = \dfrac{BC}{\sin a}\ \dfrac{\sin b}{\sin a} = \dfrac{AC}{BC} = \dfrac{BC + AB}{BC} = 1 + \dfrac{AB}{BC}$ (19-rasm).



5 -rasm.

Doira radiusini 1 ga teng deb hisoblab, biz yozishimiz mumkin $\text{И}AF = b,\ \text{И}BF = a$ Shuning uchun $AB < \text{И}AB = b - a,\ BC < BE = tga > a,$

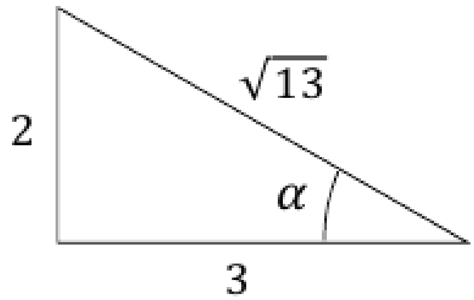
Tenglik qaraymiz: $\dfrac{\sin b}{\sin a} = 1 + \dfrac{AB}{BC}$ AB ni $a - b$ orqali, BC ni a orqali belgilaymiz.

$$\frac{\sin b}{\sin a} < 1 + \frac{b - a}{a} = \frac{b}{a}\ \text{Ю}\ \frac{\sin b}{b} < \frac{\sin a}{a}$$

Isbot tugadi.

Masala 6 Hisoblang: $2\sqrt{3}cos\left(arctg\frac{2}{3}\right)$

Yechish: Barcha teskari trigonometrik qiymatlar musbat sonlarning funksiyalari-bu 1-chorakda yotadigan burchaklar, ya’ni o‘tkir burchaklar. Shuning uchun ularni to‘g‘ri burchakli uchburchakda topish mumkin. $arctg\dfrac{2}{3}$ bu uchburchakdagi burchak, uning tangensi $\dfrac{2}{3}$ ga teng, ya’ni qarama-qarshi katetning yopishgan katetga nisbati $2:3$ ga teng. Pifagor teoremasi bo‘yicha gipotenuzani topamiz:



6 -rasm

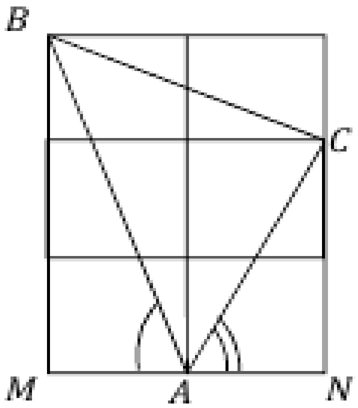
Endi arctgning har qanday triganometrik funksiyasining qiymatini topish mumkin.

$$\cos\left(arctg\frac{2}{3}\right)=\frac{3}{\sqrt{13}} \Rightarrow 2\sqrt{13}\cos\left(arctg\frac{2}{3}\right)=6$$

Javob: 6.

Masala 7. Hisoblaymiz $arctg1+arctg2+arctg3$

Yechim. $arctg3=\angle BAM,\ arctg2=\angle CAH,\ arctg1=\angle BAC$



7-rasm

$MB=3,\ AM=1,\ AB=\sqrt{10}$ va $CN=2,\ AN=1,\ AC=\sqrt{5}=BC$

$AB^2=AC^2+BC^2 \Rightarrow \angle BCA=90^{\circ},\ \angle BAC=45^{\circ}$

$arctg1+arctg2+arctg3=\pi$

Javob: π.

Mustaqil ishlash uchun misollar

1. Hisoblang $\frac{1}{\cos 20^{\circ}}-4\sin 50^{\circ}$

2. Hisoblang $\frac{2}{\sin 50^{\circ}}+8\sin 10^{\circ}$

3. Hisoblang $\frac{1}{\cos 50^{\circ}}+\frac{\sqrt{3}}{3\cos 40^{\circ}}$

4. Isbotlang $ctg70^{o} + 4\cos 70^{o} = \sqrt{3}$

5. Isbotlang $\sqrt{3}ctg20^{o} - 4\cos 20^{o} = 1$

6. Isbotlang $\frac{1}{\cos 20^{o}} - 1 = tg10^{o} \cdot tg20^{o}$

7. Isbotlang $ctg56^{o} + tg28^{o} = \frac{1}{\cos 34^{o}}$

8. Hisoblang $\sin(a - b)$, agar a va b o‘tkir burchaklar uchun $tga = \frac{3}{4}$, $tgb = 2,4$ bo‘lsa

9. Isbotlang $\arccos\frac{15}{17} - 2arcctg4 = 0$

10. Isbotlang $\arcsin\frac{4}{\sqrt{17}} + arctg4 + \arcsin\frac{8}{17} = 180^{o}$

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