MURAKKAB ARALASH TURDAGI TENGLAMA UCHUN CHEGARAVIY MASALANING QO‘YILISHI VA YAGONALIGI

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ANNOTATSIYA

Differensial tenglamalar fani rivojlanib borgan sari uni o‘rganuvchi yo‘nalishlar ham kengaya bormoqda. 30-40 yillardan boshlab o‘rganila boshlandi. Yuqori tartibli (3 va undan yuqori) tenglamalarni o‘rganishda hususiy xosilali differensial tenglamalarning asosi bo‘lgan 2-tartibli tenglamalarni tahlil etish usullaridan keng foydalanildi.

Yuqori tartibli murakkab aralash turdagi tenglamalar hayotda fizik, mexanik jarayonlarni o‘rganishda juda ko‘p uchraydi. Masalan: suyuqliklarni rezervuarlardagi harakati, mexanikada ko‘p fazoli qatlamlar matematik modellarini tuzishda yuqori tartibli tenglamalar olinadi.

Kalit so‘zlar: aralash turdagi tenglama, chegaralanmagan soha, chegaraviy yechim, regulyar yechim, trivial yechim.

Murakkab aralash turdagi tenglamalarni o‘rganish va ularni tahlil qilish usullarini yaratishda juda kup olimlar, shu jumladan A.V.Bitsadze, M.C.Salahiddinov, T.D.Jo‘rayev va ularning shogirdlari katta xissa qo‘shishgan. Shu yo‘nalishda qilingan ilmiy ishlar bilan (1-3) adabiyotlar orqali tanishish mumkin.

Ko‘rilayotgan ishda 3-tartibli murakkab aralash turdagi tenglama uchun yarim chegaralangan sohada chegaraviy masalalarni yechish bilan shug‘ullanamiz, umuman bu yo‘nalish yuqorida aytilganidan 30-40 yillardan boshlab o‘rganilayotgan bo‘lsada 2-tartibli tenglamalar yoki yuqori tartibli tenglamalar uchun yopiq sohalarda masalalar qo‘yilgan. Chegaralanmagan sohalar uchun qilingan ishlar nisbatan kam.

Ushbu ishimiz 3 qismdan iborat bo‘lib, birinchi qismida

$$\begin{cases} d\left(U_{xx}+U_{yy}\right)+C(x,y)U(x,y)=0, \\ \dfrac{d}{dx}\left(U_{xx}-U_{yy}\right)=0, \end{cases}$$

tenglama uchun chegaraviy masala qo‘yilgan.

Biz ushbu maqolada bo‘limdan ozroq izoh keltiramiz.

Eslatib o‘tamizki, murakkab turdagi tenglama deb, ham haqiqiy, ham kompleks xarakteristikalariga ega bo‘lgan tenglamaga aytiladi.

Aralash turdagi tenglama esa sohaning ma’lum qismida bir turga, ikkinchi qismida esa boshqa turga tegishli bo‘ladi. Shuning uchun bunday tenglamalar murakkab aralash turdagi tenglama deb yuritiladi.

Regulyar yechim deb, qaralayotgan sohada tenglama qatnashayotgan barcha tartibli xosilalari va tenglamani qanoatlantiruvchi funksiyaga aytiladi.

Masalaning qo‘yilishi:

Ushbu

$$\begin{cases} \frac{\partial}{\partial x}(U_{xx}+U_{yy})+C(x,y)U(x,y)=0, y>0 \\ \frac{\partial}{\partial x}(U_{xx}-U_{yy})=0, y<0 \end{cases}. \qquad (1)$$

tenglamani

$$D=\{D_1 \cup D_2 \cup y=0\}$$

sohada qaraymiz.

Bunda $D_1=\{(x,y): x>0, y>0\}$ $D_2=\{(x,y): x>0, y>-x\}$ $C(x,y)$ berilgan funksiya.

Masala. D sohada (1) tenglamaning quyidagi shartlarni qanoatlantiruvchi shartlari $U(x,y)$ yechimi topilsin:

1) $U(x,y)$ funksiya $\bar{D}$ da uzluksiz;

2) $U(x,y)$ D sohada (1) tenglamaning regulyar yechimi, $y \neq 0$;

3) $U(x,y)$ quyidagi chegaraviy shartlarni qanoatlantirsin:

$$U(0,y)=\varphi_1(y), 0 \le y < \infty \qquad (2)$$

$$U(x,-x)=\psi_1(x), 0 \le x < \infty \qquad (3)$$

$$\frac{\partial U(x,-x)}{\partial x}=\psi_2(x), 0 < x < \infty \qquad (4)$$

$$\lim_{R\to\infty} U_y=0, R^2=x^2+y^2, x>0, y>0 \qquad (5)$$

bu yerda $\varphi_i(y), \psi_i(x)$, $(i=1,2)$ berilgan funksiyalar bo‘lib, $\varphi_1(0)=\psi_1(0)$ kelishuv shartini qanoatlantiradi.

$$\frac{\partial}{\partial x}U=V \qquad (6)$$

deb belgilash kiritsak,u holda (1) tenglama

$$\begin{cases} \Delta V + CU = 0 \\ \square V = 0 \end{cases} \quad (7)$$

ko‘rinishni oladi,bunda

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\square = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$$

(2)-(4) chegaraviy shartlardan foydalanib noma’lum $V(x,y)$ funksiya uchun quyidagi chegaraviy shartlarni olamiz:

$$V(x,y)|_{y=-x} = \frac{1}{2}\left(\psi_1'(x) + \sqrt{2}\psi_2(x)\right) = \psi(x) \quad (8)$$

$$V(x,y)|_{x=0}\ \varphi_2(y) \quad (9)$$

$$\lim_{R\to\infty} V(x,y) = 0 \quad (10)$$

Masala yechimining yagonaligi

TEOREMA1. Agar (1) – (5) masala yechimga ega bo‘lsa va ushbu

a) $|C(x,y)| \le \frac{N_1}{R}$, $R \to \infty$, $N_1 = const$

b) $C_x(x,y) \ge 0$

shartlar bajarilsa, u holda masala yechimi yagona bo‘ladi.

ISBOT: Qo‘yilgan masala yechimining yagonaligini ko‘rsatish uchun bir jinsli masalani trivial yechimga ega ekanligini ko‘rsatish yetarli. Shuning uchun

$$\varphi_i(x) = \varphi_i(y) = 0 \qquad (i = 1.2) \quad (11)$$

bo‘lsin. U holda (6) belgilashga ko‘ra qo‘yilgan masala quyidagi ko‘rinishga keladi.

$$\begin{cases} \Delta V + CU = 0 \\ \square V = 0 \end{cases} \quad (7)$$

$$V(x,y)|_{y=-x} = 0 \quad (12)$$

$$V(x,y)|_{x=0} = 0$$

(13)

Masala qaralayotgan D_1 sohani R radiusli aylana bilan, D_2 esa $x - y = R$ radiusli chiziq bilan chegaralaymiz, ya’ni

$$D_{1R} = \left\{(x,y) : x^2 + y^2 < R^2, x > 0, y > 0\right\}$$

D_{2R} esa $AC: x + y = 0$ va $BC: x - y = R$ chiziqlar bilan chegaralangan soha.

D_{1R} sohaning chegarasini

$$\sigma_R = \left\{(x,y): x^2 + y^2 = R^2, x>0, y>0\right\},$$

$A(0,0)B(R,0)$ va $D(R,0)$ $A(0,0)$ chiziqlar bilan belgilaymiz.

(7) ning birinchi tenglamasini $V(x,y)$ funksiyaga ko‘paytirib, D_{1R} soha bo‘yicha integral olamiz:

$$\iint_{D_{1R}} \left[V(V_{xx}+V_{yy})+CU\right]dxdy=0 \qquad (14)$$

Bu tenglikni quyidagi ko‘rinishda yozib olamiz:

$$\iint_{D_{1R}} \left[\left(VV_x\right)_x+\left(VV_y\right)_y-\left(V_x\right)^2-\left(V_y\right)^2+\frac{1}{2}\left(CU^2\right)_x-C_xU^2\right]dxdy=0 \qquad (15)$$

Oxirgi (15) tenglikka Gauss-Ostrogradskiy formulasini qo‘llab,

$$\int_{dD_{1R}} V\left[V_x\cos(u,x)+V_y\cos(u,y)\right]ds+\frac{1}{2}\int_{dD_{1R}} CU^2dy=\iint\left[V_x^2+V_y^2+C_xU^2\right]dxdy$$

yoki

$$\int_{AB} V\frac{dv}{dn}ds+\int_{G_R} V\frac{dv}{dn}ds+\int V\frac{dv}{dn}ds+\frac{1}{2}\int_{AB} CU^2dy+\frac{1}{2}\int_{G_R} CU^2dy+\frac{1}{2}\int_{DA} CU^2dy=\iint_{D_{1R}}\left[V_x^2+V_y^2+C_xU^2\right]dxdy$$

$(C=0)$ ko‘rinishidagi ifodaga kelamiz. Bir jinsli (12) va (13) shartlardan hamda AB chiziq ustiga $dy=0$ ekanligidan foydalansak quyidagiga ega bo‘lamiz.

$$\int_{G_R} V\frac{\partial V}{\partial n}-\int_{AB} V(x,0)V(x,0)dx+\frac{1}{2}\int_{G_R} CU^2dy=\iint_{D_{1R}}\left[V_x^2+V_y^2+C_xU^2\right]dxdy \qquad (16)$$

Endi (7) ning ikkinchi tenglamasini $V(x,y)$ ga ko‘paytirib, D_{2R} soha bo‘yicha integrallaymiz.

$$\iint_{D_{2R}} V(V_{xx}-V_{yy})dxdy=0 \qquad (17)$$

Bu tenglikni quyidagi ko‘rinishda yozib olamiz:

$$\iint_{D_{2R}} (VV_x)_x-(VV_y)_y-(V_x)^2+(V_y)^2dxdy=0 \qquad (18)$$

(18) tenglikka Gauss-Ostrogradskiy formulasini qo‘llasak ushbu ifodani olamiz.

$$\int_{\partial D_{2R}} V\left[V_x\cos(n,x)+V_y\cos(n,y)\right]ds=\iint_{D_{2R}}\left[V_x^2-V_y^2\right]dxdy$$

yoki

$$\int_{AC} VV_xdy+VV_ydx+\int_{CB} VV_xdy+VV_ydx+\int_{BA} VV_xdy+VV_ydx=\iint_{D_{2R}}\left[V_x^2-V_y^2\right]dxdy \qquad (19)$$

Bir jinsli (12) va (13) shartlardan hamda AC chiziq ustida $dx=-dy$, CB chiziq ustida $dx=dy$ ekanligidan foydalanib (19) tenglikning chap tomonini hisoblaymiz.

$$\int_{CB} VV_xdy+VV_ydx=\int_{CB} V\left(V_xdy+V_ydx\right)=\int_{CB} V\left(V_xdx+V_ydy\right)=\frac{1}{2}V^2(B)$$

Bu (19) ga qo‘ysak

$$\int_{BA} VV_y dx + \frac{1}{2}V^2(B) = \iint_{D_{2R}} \left[V_x^2 - V_y^2\right] dxdy \qquad (20)$$

Hosil bo‘ladi.

Endi (20) tenglikni o‘ng tomoninihisoblaymiz. Bu ikki karrali integralni hisoblash uchun yangi harakteristik koordinatalar sistemasiga o‘tamiz. U holda D_{2R} soha A_1C_1, C_1B_1, B_1A_1 tomonli yangi sohaga o‘tadi. Bular mos holda $\eta = 0, \xi = 1$ va $\eta = \xi$ chiziqlarda yotuvchi kesmalardir. $\xi = x - y, \eta = x + y$ belgilashlarga ko‘ra $x = \frac{\xi + \eta}{2}$, $y = \frac{\eta - \xi}{2}$, $V_x = V_\xi + V_\eta$, $V_y = -V_\xi + V_\eta$ tengliklar o‘rinli bo‘ladi.

$$\iint_{D_{2R}} \left[V_x^2 - V_y^2\right] dxdy = \frac{1}{4}\iint_{D'_{2R}} \left[(V_\xi + V_\eta)^2 - (-V_\xi + V_\eta)^2\right] d\xi d\eta = \frac{1}{4}\iint_{D'_{2R}} \left[V_\xi^2 + V_\eta^2 - V_\xi^2 - V_\eta^2 + 4V_\xi V_\eta\right] d\xi d\eta$$

$$= \frac{1}{4}\iint_{D'_{2R}} 4V_\xi V_\eta d\xi d\eta = \iint_{D'_{2R}} \frac{\partial}{\partial \eta}(VV_\xi) d\xi d\eta - \iint_{D'_{2R}} VV_{\xi\eta} d\xi d\eta = \iint_{D'_{2R}} \frac{\partial}{\partial \eta}(VV_\xi) d\xi d\eta = -\int_{A_1C_1 + C_1B_1 + B_1A_1} VV_\xi d\xi$$

Bu yerda C_1B_1 chiziqda $d\xi = 0$ ekanligini e’tiborga olsak

$$\iint_{D_{2R}} \left[V_x^2 - V_y^2\right] dxdy = -\int_{A_1C_1 + B_1A_1} VV_\xi d\xi$$

Tenglikka ega bo‘lamiz.(13) va (14) shartlardan foydalanamiz.

$$\int_{A_1C_1} VV_\xi d\xi = \int_{A_1C_1} \left(\frac{1}{2}V^2\right)_\xi = \frac{1}{2}V^2\Big|_{A_1}^{C_1} = \frac{1}{2}V^2(C_1) = 0$$

$$\int_{B_1A_1} VV_\xi d\xi = \int_{B_1A_1} \left(\frac{1}{2}V^2\right)_\xi = \frac{1}{2}V^2\Big|_{B_1}^{A_1} = -\frac{1}{2}V^2(B_1)$$

Bulardan esa

$$\int_{D_{2R}} \left[V_x^2 - V_y^2\right] dxdy = -\frac{1}{2}V^2(B) \qquad (21)$$

hosil bo‘ladi. (21) ni (20) ga olib borib qo‘ysak va

$$V(x,0) = \tau_1(x)$$

$$V_y(x,0) = \nu_1(x)$$

belgilashni kiritsak,quyidagiga ega bo‘lamiz:

$$\int_{BA} VV_y = \int_B^A \tau_1(x)\nu_1(x) dx = \frac{1}{2}V^2(B) - \frac{1}{2}V^2(B) = 0$$

Bizga

$$\int_0^B \tau_1(x)\nu_1(x) dx = 0$$

ekanligi ma’lum.(10) ga asosan $R \to \infty$ da limitga o‘tamiz. Bundan esa

$$\int_0^\infty \tau_1(x)\nu_1(x)dx = 0 \qquad (22)$$

ekanligi kelib chiqadi.(22) tenglikni (16)ga qo‘yamiz va $R\to\infty$ da limitga o‘tamiz. U holda teorema shartiga ko‘ra va (13) ga ko‘ra

$$\iint_{D_1}\left[V_x^2+V_y^2\right]dxdy+\iint_{D_1}C_xU^2dxdy=0$$

tenglik o‘rinli bo‘ladi.

Agar $C_x=0$ bo‘lsa,so‘nggi tenglikdan $V_x=0$ $V_y=0$ ekanligi kelib chiqadi. Bundan esa $V=const.$

$V(0,y)=0$ shartga ko‘ra $V=U_x=0$ ga ega bo‘lamiz. Bu tenglamani $C(0,y)=0$ shart ostida yechsak, $\bar{D}_2$ sohada $U\equiv 0$ ekanligi kelib chiqadi.

Yuqoridagi tenglikdan D_1 sohada $U\equiv 0$ bo‘ladi. Funksiyaning yopiq sohada uzluksizligidan $\bar{D}_1$ sohada $U\equiv 0$ ayniyatni olamiz. $\bar{D}_1$ sohada $V\equiv 0$ ekanligidan $\tau_1\equiv 0$ tenglikka ega bo‘lamiz.

D_2 sohada (7) ning ikkinchi tenglamasi uchun Koshi masalasining yechimi yagonaligidan $V\equiv 0$ ekanligi kelib chiqadi.

$$V=U_x=0$$

Tenglamani $U(x,-x)=0$ shart bilan yechsak, $\bar{D}_2$ da $U\equiv 0$ trivial yechimga ega ekanligini ko‘rsatish mumkin.

Teorema isbotlandi.

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