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NONLINEAR REACTION-DIFFUSION EQUATIONS WITH FREE BOUNDARY CONDITIONS

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АННОТАЦИЯ

Исследуется задача со свободной границей для квазилинейного параболического уравнения с нелинейной адвекцией. Для решения задачи устанавливаются априорные оценки норм Гёльдера. На основе априорных оценок доказано существование и единственность решения.

Ключевые слова: *свободная граница, априорные оценки, существование и единственность решения.*

1. INTRODUCTION. A large number of physical, biochemical and ecological applications are associated with the reaction-diffusion problem, which is an evolutionary equation in which the spatiotemporal changes in the variable under study are due to diffusion in the spatial variable and nonlinear [1,2].

To date, various reaction-diffusion models have been constructed and studied (see [3,4]). In known works, the convection term is linear and depends only on the component density gradient [5]. However, in general, convection is also affected by the density of the components, which in turn leads to non-linear convection. In [6], the authors studied the problem with a free boundary for the reaction-diffusion equation with a nonlinear convection term. They obtained the dichotomy result and presented a constant asymptotic propagation velocity of the expanding front.

In this paper, we study the problem with a free boundary in the following formulation

$$k(u)u_t - du_{xx} - muu_x = u(a - bu), \quad (t, x) \in D, \tag{1}$$

$$u(0, x) = u_0(x), \quad 0 \le x \le s(0) \equiv s_0, \tag{2}$$

$$u_x(t, 0) = 0, \quad 0 \le t \le T, u(t, s(t)) = 0, \quad 0 \le t \le T, \tag{3}$$

$$s'(t) = -\mu u_x(t, s(t)), \quad 0 \le t \le T, \tag{4}$$

where $x = s(t)$ is the free boundary, which is defined together with the function $u(t, x)$. $k(u) \ge k_0$ for any $u > 0$, d, m, a, b, s_0, μ are positive constants. The initial function $u_0(x)$ satisfies $u_0 \in C^2([0, s_0])$, $u'_0 = u_0(s_0) = 0$, $u_0 > 0$ in $[0, s_0)$ and $\lim\limits_{x \to s_0} \frac{u_0(x)}{s_0 - x} = 0$.

In [7] author considered the free boundary problem of quasilinear reaction-diffusion equation with linear convection term (mu_x).

2.UNIQUENESS AND EXISTENCE OF THE SOLUTION

Theorem 1. Let $s(t)$, $u(t, x)$ be a solution to problem (1)-(4). Then there exist positive constants M_1, M_2 independent T such that

$$0 < u(t, x) \le M_1, \quad (t, x) \in \overline{D}, \tag{5}$$

$$0 < s'(t) \le M_2, \quad 0 \le t \le T. \tag{6}$$

Proof. The positivity of the function u(t,x) follows from the maximum principle.

To get the upper bounds, we proceed as follows. We construct an auxiliary problem for an ordinary differential equation:

$$\begin{cases} k(\overline{u})\overline{u}_t(t) = \overline{u}(a - b\overline{u}), \quad t > 0, \\ \overline{u}(0) = \|u_0\|_\infty. \end{cases}$$

Its solution is given by the following explicit formula [7]:

$$\overline{u}(t) = \frac{e^{\int_0^t \alpha(\eta) d\eta}}{\int_0^t \beta(\eta) \left(\int_0^\eta e^{\alpha(\xi) d\xi}\right) d\eta + \frac{1}{\|u_0\|_\infty}},$$

where $\alpha(t) = \frac{a}{k(\overline{u}(t))}$, $\beta(t) = \frac{b}{k(\overline{u}(t))}$. Comparing $u(t, x)$ with $\overline{u}(t)$ yields that

$$u(t, x) \leqslant \sup_{t \geqslant 0} \overline{u}(t) \equiv M_1.$$

Further, taking into account the condition (4) and the positivity of the function $u(t,x)$ in the domain D, we find $u_x(t,s(t)) < 0$. Therefore, we get $s'(t) > 0$.

To set an upper bound for $s'(t)$, in problem (1)-(4), replacing

$$U(t,x) = u(t,x) + N(x - s(t)) \tag{7}$$

and we get the problem with respect to $U(t,x)$

$$\begin{cases} k(U)U_t - dU_{xx} - muU_x \le u(a - mN) \le 0, & (t,x) \in D, \\ U(0,x) = u_0(x) + N(x - s_0) \le 0, & 0 \le x \le s_0, \\ U_x(t,0) = N, \quad U\big(t,s(t)\big) = 0, & 0 \le t \le T. \end{cases}$$

By choosing $N \ge \max\left\{\frac{a}{m}, \max\limits_{x}\left|\frac{u_0}{s_0 - x}\right|\right\}$ in $\overline{D}$, then we have $U(t,x) \le 0$. Therefore, taking into account (7), we find $u_x(t,s(t)) \ge -N$, which is equivalent to $s'(t) \le \mu N$. The proof is complete.

First, we obtain a priori estimates for $u(t,x)$. Due to the boundary conditions (3)-(7), we cannot use the results of the work [8]. Therefore, we introduce the following transformation

$$t = t, \quad y = \frac{x}{s(t)}$$

to straighten the free boundary. Then the domain D is transformed into the domain $D_1 = \{(t,y): 0 < t < T, 0 < y < 1\}$, and the bounded function $U(t,y) = u(t,x)$ is a solution to the problem

$$U_t - A(t,y,U)U_{yy} = B(t,y,U,U_y), \quad (t,y) \in D_1, \tag{8}$$

$$U(0,y) = U_0(y) \equiv u_0(s_0 y), \quad 0 \le y \le 1, \tag{9}$$

$$U_y(t,0) = 0, \quad U(t,1) = 0, \quad 0 \le t \le T, \tag{10}$$

where $D_1 = \{(t,y): 0 < t \le T, \quad 0 < y < 1, A(t,y,U) = \frac{d}{k(U)s^2(t)}$, $B(t,y,U,U_y) = \frac{f(U)}{k(U)} + \frac{ys\prime(t) + mU}{k(U)s(t)}U_y$, $s'(t) = -\frac{\mu}{s(t)}U_y(t,s(t))$.

Theorem 2. Suppose that a function $U(t,y)$ continuous in D_1 satisfies the conditions of problem (8)-(10) Assume that, for $(t,y) \in D_1$, $|U| \le M_1$ and any U_y, continuous functions $A(t,y,U)$ and $B(t,y,U,U_y)$ satisfy the condition

$$\frac{|B(t,y,U,U_y)|}{A(t,y,U)} \le K(U_y^2+1), \quad K>0. \tag{11}$$

Under the conditions (11), the following estimate is valid

$$|U_y(t,y)| \le M_3(M_1, M_2, \min A, K, \delta), \quad (t,y) \in D_1^{\delta}.$$

In addition, in the domain $\{(t,y) \in \overline{D}_1, |U| \le M_1, |U_y| \le M_3\}$ we have the estimate

$$|U|_{\frac{2}{3}}^{D_1^{2\delta}} \le M_4(M_1, M_2, \frac{d}{s_0^2 k_0}, K, \delta).$$

And if it is also known that $U(t,y)$ possesses in $\overline{D}_1$ summable with a square generalized derivatives U_{ty}, U_{yy}, then

$$|U|_{1+\alpha}^{D_1^{2\delta}} \le M_5(M_1, M_2, M_3, \frac{d}{s_0^2 k_0}, K, \delta), \quad 0<\alpha<1.$$

If $U|_{\Gamma(t=0,y=0,y=1)} = 0$, then the estimates are also valid in $\overline{D}_1$; where $\Gamma(t=0, y=0, y=1)$ is a parabolic boundary, $D_1^{\delta} = \{(t,y): 0<\delta \le t \le T,\ \delta \le y \le 1-\delta\}$.

Since the estimates $|u(t,x)| \le M_1$, $|s'(t)| \le M_2$, have been established, the estimates of Theorem 2 are obtained by virtue of Theorem 3 in [8].

Theorem 3. Under assumption of Theorems 1 and 2, problem (1)-(4) has a unique solution.

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