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PREPARATION FOR MATHEMATICS OLYMPIADS FOR UNIVERSITY STUDENTS

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ABSTRACT:

This article presents a sample mock exam to prepare for Mathematical Olympiads. It presents complex problems and their solutions from fields such as algebra, mathematical analysis, topology, combinatorics, number theory. Necessary for solving given problems, important theorems and affirmations are given. Modern and unusual methods for solving mathematical problems are described. For each problem, a marking scheme is also given that complies with the requirements of International Mathematical Olympiads.

Keywords: *Cauchy-Buniakowski inequality, fixed point, idempotent matrices, Sylvester rank inequality, sequence, Darboux sum, Bolzano-Weierstrass theorem, limit point, prime number, field, invertible, pigeonhole principle, Cayley-Hamilton theorem, characteristic polynomial, eigenvalues, algebraic closure, continuous function, Euclidean plane, uncountable set, neighbourhood, injective, surjective, bijective, simple graph, planar graph, subgraph, Kuratowski's Theorem, Euler's formula, metric space.*

At the Olympiads in mathematics, mainly unconventional problems are used. Among them is the olympiads of mathematics for students. In this, in addition to general mathematical knowledge, the participant will need skills and qualifications such as creativity, logical thinking, a theorem corresponding to a given issue, a lemma, the ability to identify affirmations. Among the most famous of the Mathematical Olympiads for university students are IMC (Bulgaria), AKHIMO (Uzbekistan), NCUMC (Russia), OMOUS (Turkmenistan), RUDN MATH OLYMP (Russia), NMC (Russia), SEEMOUS (Europe) and others. Below we bring a sample mock exam for students to prepare for these Olympiads, as well as an marking scheme for each problem.

MOCK EXAM FOR MATHEMATICS OLYMPIADS FOR UNIVERSITY STUDENTS

Problem 1. If $a_i > 0$ for all $i \in \{1,2,\ldots,n\}$ then, prove that:

$$\left(\frac{a_1}{a_2}\right)^{2023} + \left(\frac{a_2}{a_3}\right)^{2023} +\ldots+ \left(\frac{a_n}{a_1}\right)^{2023} \geq \frac{a_1}{a_2} + \frac{a_2}{a_3} +\ldots+ \frac{a_n}{a_1}$$

Solution: We define $x_i = \frac{a_i}{a_{i+1}}$ for all $i \in \{1,2\ldots n\}$ where $a_{n+1} = a_1$, then $x_1 \cdot x_2 \cdot\ldots\cdot x_n = 1$. (1.1). By Cauchy-Buniakowski inequality we know that:

$$x_1^{2023} + x_2^{2023} +\ldots+ x_n^{2023} \geq \frac{(x_1 + x_2 +\ldots+ x_n)^{2023}}{n^{2022}} =$$

$$= (x_1 + x_2 +\ldots+ x_n) \cdot \left(\frac{x_1 + x_2 +\ldots+ x_n}{n}\right)^{2022} \geq (x_1 + x_2 +\ldots+ x_n)$$

Hence:

$$\left(\frac{a_1}{a_2}\right)^{2023} + \left(\frac{a_2}{a_3}\right)^{2023} +\ldots+ \left(\frac{a_n}{a_1}\right)^{2023} \geq \frac{a_1}{a_2} + \frac{a_2}{a_3} +\ldots+ \frac{a_n}{a_1}$$

Marking scheme:

(a) If shown to be (1.1)..+2 points.

(b) For any complete proof ....10 points.

Problem 2. Prove that, given $f: [0; 1] \to [0; 1]$ a continuous function has at least one fixed point.

Solution: We define $g(x) := f(x) - x$, this function is coninuous in the segment $[0; 1]$. We have the following two cases:

(i) $f(1) = 1$ or $f(0) = 0$ then f has at least one fixed point.

(ii) The first case is not appropriate, then $f(1) < 1$ and $f(0) > 0$. Hence, $g(0) > 0$ and $g(1) < 0$. From $g(0) \cdot g(1) < 0$, (2.1) there is at least one point $c \in (0; 1)$, such that $g(c) = 0$, because g is continuous. This c be fixed point for f function. (2.2)

Hence, given $f: [0; 1] \to [0; 1]$ a continuous function has at least one fixed point.

Marking scheme:

(a) If function g is chosen ....+3 points.

(b) If it is shown that g is continuous ....+1 points.

(c) If (2.1) is proved ....+4 points.

(d) If shown to be (2.2)..+2 points.

Problem 3. Let $A_1, A_2, \dots, A_k$ be idempotent matrices ($A_i^2 = A_i$) in $M_n(\mathbb{R})$. Prove that

$$\sum_{i=1}^{k} N(A_i) \geq rank\left(I - \prod_{i=1}^{k} A_i\right)$$

where $N(A_i) = n - rank(A_i)$ and $M_n(\mathbb{R})$ is the set of square $n \times n$ matrices with real entries.

Solution: Fix an index i and note that because A_i is idempotent, then $A_i(I_n - A_i)X_i = O_n$ for any matrix X_i. Hence, by Sylvester rank inequality:

$$n - rk(A_i) \geq rk\big((I_n - A_i)X_i\big) - rk(O_n) = rk((I_n - A_i)X_i\ ). \qquad (3.1)$$

for any matrix X_i. By summing and using the supra-additivity ($rk(A) + rk(B) \geq\geq rk(A + B)$) of the rank we may infer

$$\sum_{i=1}^{k}\big(n - rk(A_i)\big) \geq \sum_{i=1}^{k} rk\big((I_n - A_i)X_i\big) \geq rk\left(\sum_{i=1}^{k}(I_n - A_i)X_i\ \right). \qquad (3.2)$$

By letting $X_1 = I_n$ and $X_i = A_1 A_2 \dots A_{i-1}$ for $i \geq 2$ the conclusion immediately follows.

$$\sum_{i}^{k} N(A_i) \geq\ rank\left(I - \prod_{i}^{k} A_i\right)$$

Note: Sylvester rank inequality:

$$rank(A) + rank(B) \leq rank(AB) + n$$

where $A, B \in M_n(\mathbb{R})$.

Marking scheme:

(a) If (3.1) is proved ....+3 points.

(b) If (3.2) is proved ....+3 points.

(c) If X_i are chosen and proof is complete..............................+4 points.

Problem 4. Let $s_n = \int_0^1 sin^n\,(nx)dx$.

(i) Prove that $s_n \leq \frac{2}{n}$ for all odd n.

(ii) Find all the limit points of the sequence $s_1, s_2, s_3, \ldots$.

Solution: We proceed with each part as follows:

(i) By the substitution $x \mapsto \frac{x}{n}$ may consider $s_n = \frac{1}{n}\int_0^1 sin^n(x)dx$. Note that because $sin(x) = -sin(x+\pi)$, we have $sin^n(x) = -sin^n(x+\pi)$ for odd n. Let $k \equiv n \ (mod\ 2\pi)$. By the parity of $sin^n(x)$, we know that $\int_0^{2\pi} sin^n(x)dx = 0$. Hence,

$$\frac{1}{n}\int_0^n sin^n(x)dx = \frac{1}{n}\int_0^k sin^n(x)dx \leq \frac{1}{n}\int_0^\pi sin^n(x)\,dx = \frac{2}{n}$$

as required.

(ii) We shall prove that the only limit point is 0. We already know that $s_n \to 0$ for odd n, so it suffices to prove that $s_n \to 0$ for even n. To do so, we will bound the function from above using rectangles, i.e. we shall take an appropriate upper Darboux sum to bound $\int_0^1 sin^n(x)dx$.

For a sufficiently small $\delta > 0$, we define the partition

$$P_\delta(n) = \left\{\left[0;\frac{\pi}{2}-\delta\right),\left[\frac{\pi}{2}-\delta;\frac{\pi}{2}+\delta\right),\left[\frac{\pi}{2}+\delta;\frac{3\pi}{2}-\delta\right),\ldots,\left[\frac{X\pi}{2}\pm\delta;n\right]\right\}$$

For each interval in $P_\delta(n)$ containing some multiple of $\frac{\pi}{2}$, we bound $sin^n(x)$ by a rectangle of height 1. Otherwise, we can bound the interval by $sin^n(x)$ by a rectangle of height $sin^n\left(\frac{\pi}{2}-\delta\right) = cos^n(\delta)$, where we note that $cos^n(\delta) \to 0$ as $n \to \infty$. We note that there are less than n multiples of $\frac{\pi}{2}$ in the interval $[0;\ n]$, so the sum of the areas of these rectangles is at most $2n\delta + \ ncos^n(\delta)$, and hence

$s_n = \frac{1}{n}\left(2n\delta + \ ncos^n(\delta)\right) = 2\delta + cos^n(\delta) \to 2\delta,$ as $n \to \infty.$

Now suppose for the sake of contradiction that $2\varepsilon > 0$ was a limit point of the sequence. By selecting $\delta < \varepsilon$, we show that the terms of s_n are eventually inside the interval $(0;\ 2\varepsilon)$, a contradiction. Then the sequence is bounded in the interval $[0;\ 1]$ but has no limit point greater than 0, hence by Bolzano-Weierstrass theorem, the only limit point is 0.

Marking scheme:

(a) If part (i) is proved ....+4 points.

(b) If part (ii) is proved ....+6 points.

(c) For any complete proof ....10 points.

Problem 5. Let $p > 3$ be a prime number. A sequence of $p-1$ integers $a_1, a_2, \ldots, a_{p-1}$ is called *wonky* if they are distinct modulo p and $a_i a_{i+2} \not\equiv a_{i+1}^2\ (mod\ p)$ for all $i \in \{1, 2, \ldots, p\ -1\}$, where $a_p = a_1$ and

$a_{p+1} = a_2$. Does there always exist a *wonky* sequence such that

$$a_1a_2,\ \ a_1a_2 + a_2a_3,\qquad \ldots,\qquad a_1a_2 + \ldots + a_{p-1}a_1$$

are all distinct modulo p?

Solution: Throughout this solution all congruences are taken modulo p. Our construction will be $a_i \equiv \frac{1}{i}\ (mod\ p)$ (5.1). We now verify this construction works. Note that:

$$a_i \equiv a_j \Rightarrow \frac{1}{i} \equiv \frac{1}{j} \Rightarrow i \equiv j \Rightarrow i = j$$

Then if $k < p-1$,

$$\sum_{j=1}^{k} a_j a_{j+1} \equiv \sum_{j=1}^{k} \frac{1}{j(j+1)} \equiv \sum_{j=1}^{k} \left(\frac{1}{j} - \frac{1}{j+1}\right) \equiv 1 - \frac{1}{k+1} \qquad (5.2)$$

which are all distinct since

$$1-\frac{1}{k+1}\equiv 1-\frac{1}{m+1}\Rightarrow \frac{1}{k+1}\equiv \frac{1}{m+1}\Rightarrow k=m \qquad (5.3)$$

Moreover

$$\sum_{j=1}^{p-1} a_j a_{j+1}\equiv \sum_{j=1}^{p-2} a_j a_{j+1}+a_{p-1}a_1\equiv 1-\frac{1}{p-1}-1\equiv 1,$$

and $1-\frac{1}{k+1}\neq 1\ \forall\, k$ so this is distinct from all the previous terms (5.4). Finally if $i<p-2$:

$$a_i a_{i+2}\equiv \frac{1}{i(i+2)}\neq \frac{1}{i(i+2)+1}\equiv \frac{1}{(i+1)^2}\equiv\ a_{i+1}^2$$

Additionally if $i\in\{p-2;p-1\}$, $a_i a_{i+2}\equiv -\frac{1}{2}\neq 1$ since $p\neq 3$ (5.5).

Marking scheme:

(a) If (5.1) is chosen ....+3 points.

(b) If (5.2) is proved ....+3 points.

(c) If (5.3) is proved ....+1 points.

(d) If (5.4) is proved ....+1 points.

(e) If (5.5) is proved ....+2 points.

Problem 6. Let A be a square matrix with entries in the field $\mathbb{Z}/p\mathbb{Z}$ such that A^n-I is invertible for every positive integer n. Prove that there exists a positive integer m such that $A^m=0$.

Note: A matrix having entries in the field $\mathbb{Z}/p\mathbb{Z}$ means that two matrices are considered the same if each pair of corresponding entries differ by a multiple of p.

Solution 1: Note that there are finitely many matrices under consideration. Let the number be $k\in\mathbb{N}$. Then by pigeonhole principle, two of the $k+1$ matrices $A,A^2,A^3,\ldots,A^{k+1}$ must be equal. Suppose that they are A^r and A^s, where $r>s$. Then note that:

$$A^r - A^s = 0 \Leftrightarrow A^s(A^{r-s} - I) = 0 \Leftrightarrow A^s = 0, \qquad (6.1)$$

since $A^{r-s} - I$ is invertible.

Marking scheme:

(a) If it shows that the number of matrices is finite ....+2 points.

(b) If the pigeonhole principle is used ....+3 points.

(c) If (6.1) is proved ....+5 points.

Solution 2: We shall prove the statement using eigenvalues and the Cayley-Hamilton theorem.

By the Cayley-Hamilton theorem, the characteristic polynomial c_A of A satisfies $c_A(A) = 0$.

Consider the eigenvalues of A over the algebraic closure of $\mathbb{Z}/p\mathbb{Z}$. The closure is countable, but all elements have finite order. Suppose for the sake of contradiction that some eigenvalue λ is not zero. Then there exists $m \in \mathbb{N}$ such that $\lambda^m = 1$. However this implies $det(A^m - I) = 0$, and $A^m - I$ is therefore not invertible, which is a contradiction.

Since all eigenvalues are 0, the characteristic polynomial must be $c_A = X^n$. Thus, $c_A(A) = A^n = 0$, as required.

Problem 7. Let S be a set with 10 distinct elements. A set T of subsets of S (possibly containing the empty set) is called *union-closed* if, for all $A, B \in T$, it is true that $A \cup B \in T$. Show that the number of *union-closed* sets T is less than 2^{1023}.

Solution: Let $f(n)$ denote the number of union-closed sets T over a set $S_n = \{1, 2, \dots, n\}$. We bound $f(10)$ by showing that $f(n) \leq f(n-1)^2$.

Let T be a *union-closed* set over S_n, for $n \geq 1$. We partition T into A and B, where $A = \{t \in T : n \notin t\}$, and $B = \{t \in T : n \in t\}$. Then define $B' = \{b \setminus \{n\}: b \in B\}$. Note that both A and B' are *union-closed* sets over S_{n-1}, and furthermore, T uniquely partitions into A and B' by the above method. Since there are $f(n-1)^2$ choices of A and B', we have $f(n) \leq f(n-1)^2$ (7.1), as desired (we have an inequality here since not all T generated by arbitrary

A and B' will be union-closed over S_n).

We now count that $f(2) \leq 14$, since $\left|P\left(P(S_2)\right)\right| = 16$ (7.2), but $\{\{1\},\{2\}\}$ and $\{\emptyset,\{1\},\{2\}\}$ are not *union-closed*. Hence $f(10) \leq f(2)^{2^8} = 14^{256} < 2^{1023}$, since $\left(\frac{14}{16}\right)^{256} < \frac{1}{2}$.

Marking scheme:

(a) If A and B sets are chosen ....+2 points.

(b) If (7.1) is proved ....6 points.

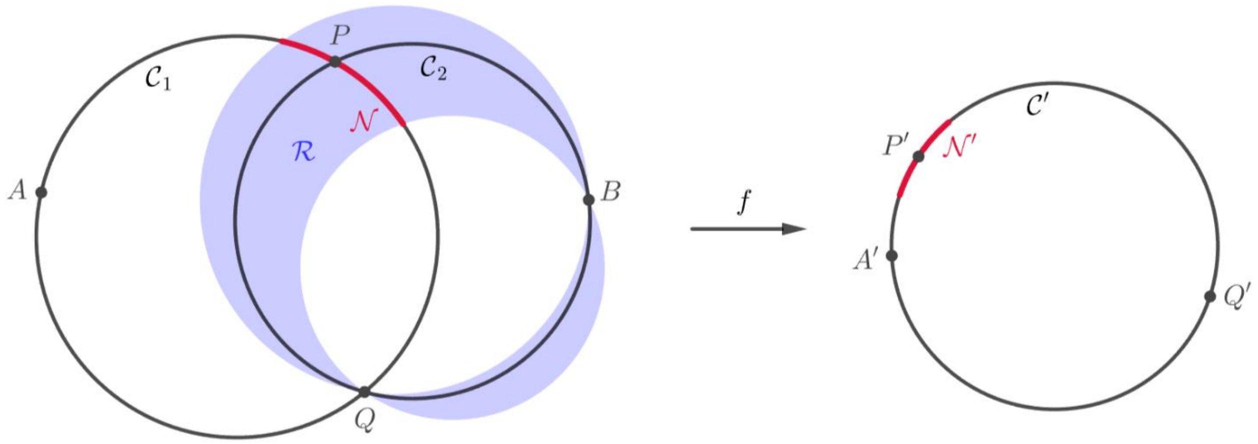
(c) If (7.2) is calculated ....+2 points.

(d) If the proof is completed correctly ....+2 points.

Problem 8. Let $\mathbb{R}^2$ denote the Euclidean plane. A continuous function $f: \mathbb{R}^2 \to \mathbb{R}^2$ maps circles to circles. (A point is not a circle.) Prove that it maps lines to lines.

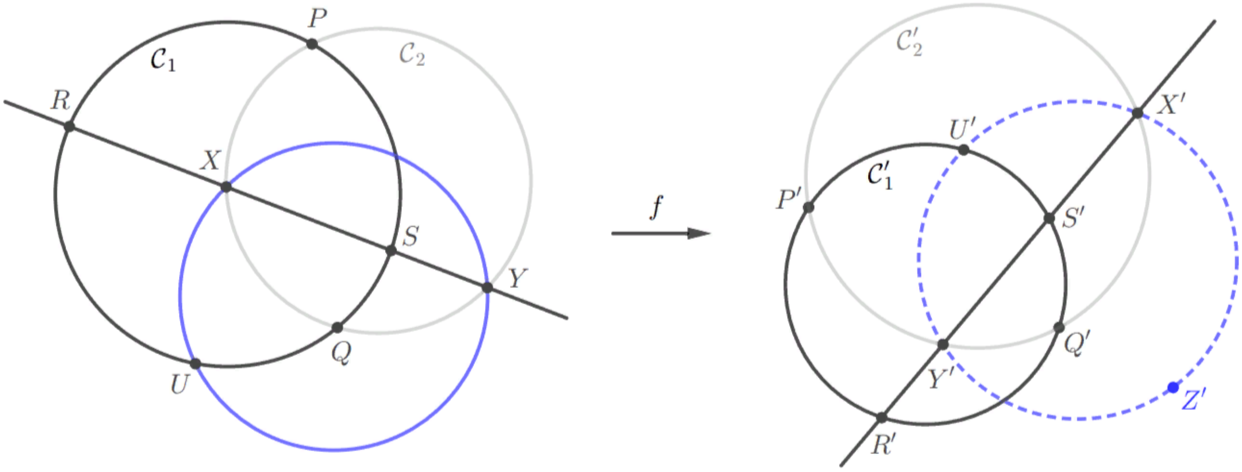
Solution: In this proof, we use the notation $X' := f(X)$. The proof can be broken down into four main steps.

***f* is injective:** Suppose that A and B are distinct points such that $A' = B'$. Since the range of the function is at least a circle, which is an uncountable set, there must exist points P and Q so that A, B, P, Q lie in general position, and A', P', Q' are distinct. Let $\mathcal{C}_1$ be the circle passing through APQ and $\mathcal{C}_2$ be the circle passing through BPQ. Note that $\mathcal{C}_1'$ and $\mathcal{C}_2'$ both map to the circumcircle $\mathcal{C}'$ of $A'P'Q'$, which must exist.



Now, since f is continuous, there exists a neighbourhood $\mathcal{N}$ of P on $\mathcal{C}_1$ which maps to a subset of $\mathcal{C}' \setminus \{A'; Q'\}$. Consider a perturbation of the circle $\mathcal{C}_2$, such that it still passes through B and Q, but now passes through a point in $\mathcal{N}$ distinct from P. Each of these circles in the perturbation must then also map to $\mathcal{C}'$ by the same argument as above. Hence, this perturbation creates a region $\mathcal{R}$ such that $f(\mathcal{R}) = \mathcal{C}'$. However, note that any circle contained entirely within $\mathcal{R}$ must then also map to exactly $\mathcal{C}'$. This implies that there are arbitrarily small circles which map to $\mathcal{C}'$, contradicting the continuity of f. Hence f is injective.

***f* is surjective:** Since f is injective, we now know that $\mathcal{C}_1'$and $\mathcal{C}_2'$ must intersect only at P' and Q'. Let $X, Y \in \mathcal{C}_2$ so that $\mathcal{C}_1$ separates X from Y. By injective continuity we can deduce that $\mathcal{C}_1'$ separates $arc\ P'X'Q'$ from $arc\ P'Y'Q'$, and hence X' from Y'. Let the intersection of $X'Y'$ with $\mathcal{C}_1'$ be R' and S'. WLOG, assume that S is not an intersection of XY with $\mathcal{C}_1$, then the circumcircle of XYS must map to the circumcircle of $X'Y'S'$, a contradiction. Hence R and S must be the intersection of XY with $\mathcal{C}_1$.

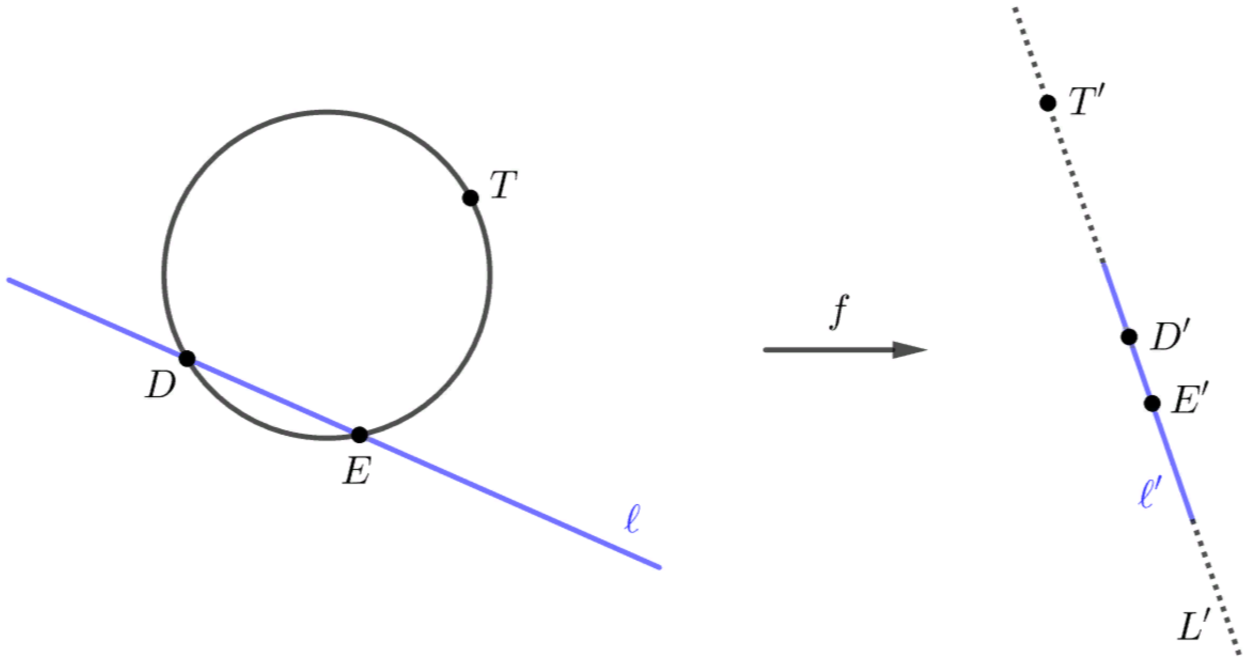


Consider any point Z' not on the line $X'Y'$. Let the circumcircle of $X'Y'Z'$ intersect $\mathcal{C}_1'$ at U'. We know this intersection point exists because the circumcircle of $X'Y'Z'$ passes through X' and Y', which are separated by $\mathcal{C}_1'$. Since f is injective, U is distinct from S and R, and hence the circumcircle of XYU which must map to the circumcircle of $X'Y'U'$, which passes through Z'. Hence any point Z' not on the line $X'Y$' is in the image of f.

Repeating this argument replacing X with a different point W on the $arc\ PXQ$ shows that every point is in the image of f. This proves that f is surjective and thus bijective.

***f* preserves collinearity:** Above, we proved that the family of circles passing through X' and Y' is mapped to by some circle passing through X and Y. Conversely, any circle passing through X and Y must map to a circle passing through X' and Y'. Hence, the family of circles passing through X and Y maps to the family of circles passing through X' and Y'. Since f is bijective the complement of the family must map to the complement of the image family. This shows that for any point T on the line XY , T' must lie on the line $X'Y'$.

***f* maps lines to lines:** Finally, since f is bijective and preserves collinearity, a line ℓ must map to a subset ℓ' of a line L'. Let $T' \in L'$. If $T \notin \ell'$, then T must lie off the lie ℓ. Consider a circle passing through T and two points $D, E \in \ell$. This must map to a circle, but $T'D'E'$ forms a line, a contradiction. Hence, lines must map to lines.



Marking scheme:

(a) If it is proved that f is injective ....+3 points.

(b) If it is proved that f is surjective ....+3 points.

(c) If it is proved that f preserves collinearity ....+2 points.

(d) If it is proved that f maps lines to lines ....+2 points.

Problem 9. Let G be a simple graph with n vertices and m edges such that no two cycles share an edge. Prove that $2m < 3n$.

Note: A simple graph is a graph with at most one edge between any two vertices and no edges from any vertex to itself. A *cycle* is a sequence of distinct vertices $v_1, \ldots, v_n$ such that there is an edge between any two consecutive vertices, and between v_n and v_1.

Solution 1: Every cycle contains ≥ 3 edges, but every edge is in ≤ 1 cycle. Denote the number of cycles by C, then we have $C \leq \frac{m}{3}$ (9.1). Delete an edge from each cycle to obtain the graph G'. G' has no cycles, so it has at most $n-1$ edges. (This is a well-known fact.) Then $m = |E(G)| = C + |E(G')| \leq \frac{m}{3} + n - 1 << \frac{m}{3} + n$, and hence $\frac{2m}{3} < n$ as required (9.2).

Marking scheme:

(a) If (9.1) is proved ....+2 points.

(b) If G‘ is chosen ....+3 points.

(c) If (9.2) is proved ....+5 points.

Solution 2: G must be planar. This can be most quickly shown using *Kuratowski's Theorem*. Suppose G is not planar, then it contains a subgraph $K_{3,3}$ or K_5 , which contradicts the assumption that cycles in G do not share an edge.

Therefore G is planar, so every cycle contributes 1 to the number of faces. Since the number of cycles C satisfies $C \leq \frac{E}{3}$ (9.1), and $F = C + 1$, applying *Euler's* formula $F - E + V = 2$ completes the proof (9.3).

Note: *(Kuratowski's Theorem:)* A graph is planar if and only if it does not contain any subdivision of $K_{3,3}$ or K_5.

Marking scheme:

(a) If (9.1) is proved ....+2 points.

(b) If (9.3) is proved ....+3 points.

(c) If it is proved that G is planar ....+5 points.

(d) If it is not shown that G is planar ....-1 points.

Problem 10. Let $(X;\ d)$ be a nonempty connected metric space such that the limit of every convergent sequence, is a term of that sequence. Prove that X has exactly one element.

Solution: Suppose $|X| > 1$. Take $x \in X$ and consider $A = \{x\}, B = X - \{x\}$.(so B becomes non empty) Since X is connected we must have either $cls(A) \cap B \neq \phi$ or $cls(B) \cap A \neq \phi$. But $cls(A) = A$ and so $cls(A) \cap B = A \cap B = \phi$. So we must have $cls(B) \cap A \neq \phi \Rightarrow x \in cls(B)$ (10.1). So every $x \in X$ is a limit point of X. So for all $\varepsilon > 0$ we must have $X \cap B_\varepsilon(x)$ is infinite. So exist a convergent sequence $\{x_n\}$ going to x and none of x_n equals to x. Now for any small $\varepsilon > 0$ exist $x \neq x_{n_\varepsilon}$ such that $d(x, x_{n_\varepsilon}) < \varepsilon$. Now we know for all such $n\_\varepsilon, x_{n_\varepsilon}$ is a limit point of X. Now any subsequence of $\{x_n\}$ can't go to x_{n_ε} so exist $y_{n_\varepsilon} \neq x_{n_\varepsilon}$ and not equals to any term of the sequence $\{x_n\}$ such that $d(x_{n_\varepsilon}, y_{n_\varepsilon}) < \varepsilon \Rightarrow d(x, y_{n_\varepsilon}) < 2\varepsilon$ (10.2). So we get a sequence $\{y_{n_\varepsilon}\}$ going to x while no terms of $\{y_{n_\varepsilon}\}$ is a term of sequence $\{x_n\}$. Now as per given condition we know x is a term of $\{y_{n_\varepsilon}\}$ as well as of $\{x_n\}$ but that's impossible (10.3). So, $|X| = 1$. Here $cls(S)$ denotes closure of set S.

Marking scheme:

(a) If (10.1) is proved ....+3 points.

(b) If it is shown that x_{n_ε} ....+1 points.

(c) If y_{n_ε} is chosen ....+2 points.

(d) If (10.2) is proved ....+2 points.

(e) If (10.3) is proved ....+2 points.

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